Magnetic Dipoles - General Current Distribution

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Here we consider the multipole expansion of the vector potential due to a generic current distribution. The goal is to calculate the expansion up to the dipole term, showing that there is no monopole contribution. As in the simpler case of a circular current, one starts by expanding the denominator in the integral expression for the vector potential.

$$\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|} = \frac{\mu_0}{4\pi}\int d^3r' \, \bar{J}(\bar{r}')\left(\frac{1}{r} + \frac{\bar{r}\cdot\bar{r}'}{r^3} + \dots\right)$$

Monopole term

In order to show that the first term vanishes, one observes that (using components notation)

$$\partial_j (J_j r_i) = (\partial_j J_j)\, r_i + J_j \delta_{ij} = \underbrace{\nabla\cdot\bar{J}}_{=0} \, r_i + J_i = J_i$$

summed indices

continuity eq.

Consequently, one can rewrite the first term of the expansion as follows (in components)

$$A_i(\bar{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{J_i(\bar{r}')}{r} = \frac{\mu_0}{4\pi}\int d^3r' \, \underbrace{\partial'_j\left(\frac{J_j r'_i}{r}\right)}_{\text{total derivative}}$$

The integrals of a total derivative can be evaluated with the Gauss theorem. If the current density is localized, the integral over the surface of the gaussian region, chosen to be equal to R^3 (all space) is zero.

Dipole term

In order to deal with the second term it is convenient to start from the identity

$$\partial_j \left(J_j r_i r_k \right) = \left(\partial_j J_j \right) r_i r_k + J_i r_k + J_k r_i = J_i r_k + J_k r_i$$

The above can be used to write

$$J_i r_k = \partial_j \left(J_j r_i r_k \right) - J_k r_i$$

Therefore

$$\int d^3r' \; J_i(r')\, r_k\, r_k' = \int d^3r' \; \frac{r_k}{2} \left(J_i r_k' + J_i r_k' \right)$$

$$= \int d^3r' \; \frac{r_k}{2} \left[J_i r_k' + \underbrace{\partial_j' \left(J_j r_i' r_k' \right)}_{\text{integrates to zero}} - J_k r_i' \right]$$

$$= \int d^3r' \; \frac{r_k}{2} \left[J_i r_k' - J_k r_i' \right] = \frac{1}{2} \int d^3r' \left[J_i \, \bar{r}\cdot\bar{r}' - \bar{r}\cdot\bar{J}\, r_i' \right]$$

By using the vector identity

$$\bar{A} \times \left(\bar{B} \times \bar{C} \right) = \bar{B} \left(\bar{A}\cdot\bar{C} \right) - \bar{C} \left(\bar{A}\cdot\bar{B} \right)$$

One can rewrite the integral as

$$\int d^3r' \; J_i \bar{r}\cdot\bar{r}' = \frac{1}{2} \int d^3r' \left[J_i \left(\bar{r}\cdot\bar{r}' \right) - r_i' \left(\bar{J}\cdot\bar{r} \right) \right] = \frac{1}{2} \int d^3r' \left[\overset{\bar{A}}{\bar{r}} \times \left(\overset{\bar{B}}{\bar{J}} \times \overset{\bar{C}}{\bar{r}'} \right) \right]_i$$

$$= \bar{r} \times \underbrace{\frac{1}{2} \left(\int d^3r' \; \bar{J}(\bar{r}') \times \bar{r}' \right)}_{\equiv -\bar{m}}$$

So that finally

$$\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi} \; \frac{\bar{m} \times \bar{r}}{r^3}$$

We now need to check that this generic definition of the magnetic moment is consistent with the definition $m = I\,S$ in the simpler case of a circular loop of current. If the circle has radius a and lies on the x-y plane

$$\bar{J}(\bar{r}') = \frac{I}{a}\,\delta(r-a)\,\delta\left(\vartheta - \frac{\pi}{2}\right)\hat{\varphi}$$

$$\bar{m} = \frac{1}{2}\int d^3r'\, \bar{r}' \times \bar{J}(\bar{r}') = I\,\frac{\hat{k}}{2}\int_0^{2\pi} d\varphi\, a^2 = \pi a^2 I\,\hat{k}$$

$$= S\,I\,\hat{k}$$

Q. E. D.