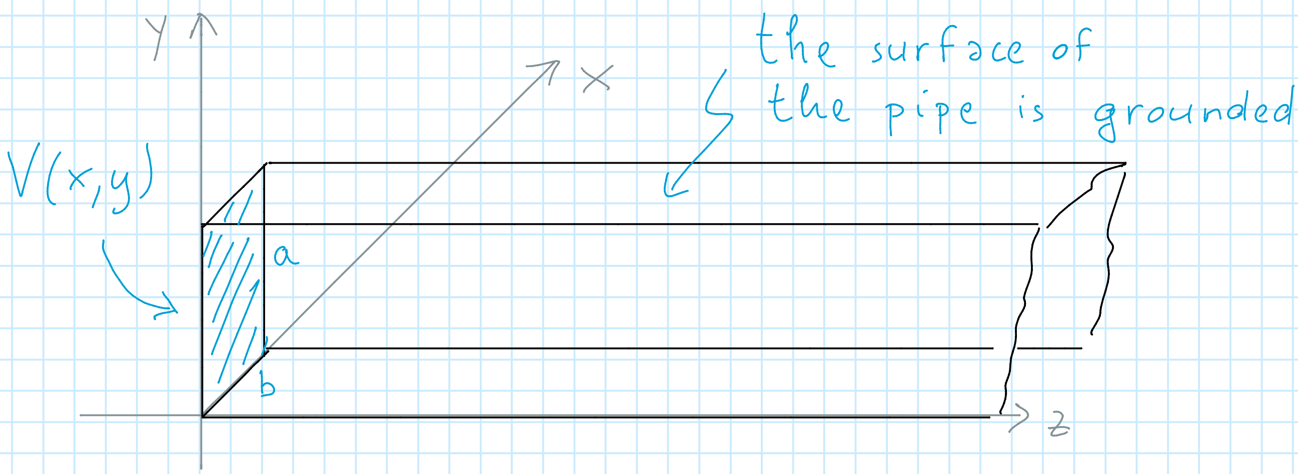
Separation of variables - Cartesian 3D

Sunday, January 6, 2019 4:01 AM

Here we want to apply the separation of variables to a case in which all of the three cartesian coordinates play a role. We consider a pipe of rectangular section closed on one side and infinitely long in the other direction. The potential on the side wall of the pipe is zero, while the potential on the surface which closes off the pipe is assumed to be a known function of x and y.



The Laplace equation satisfied by the potential inside the pipe is

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

The boundary conditions are

1) $\varphi(0, y, z) = 0$ for $z > 0$

2) $\varphi(b, y, z) = 0$ for $z > 0$

3) $\varphi(x, 0, z) = 0$ for $z > 0$

4) $\varphi(x, a, z) = 0$ for $z > 0$

5) $\varphi \to 0$ for $z \to \infty$

6) $\varphi(x, y, 0) = V(x,y)$ where V is a known function

We look for a factored solution of the form

$$\varphi(x, y, z) = X(x)\,Y(y)\,Z(z)$$

Therefore

$$\frac{1}{\varphi}\,\Delta\varphi = 0 \longrightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

Consequently it must be

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_x \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = C_y \qquad \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = C_z$$

with C_x, C_y, C_z constants such that

$$C_x + C_y + C_z = 0$$

Because of the boundary condition 5, one needs to choose a positive C_z, so that one can have a decaying exponential as the solution for Z

$$C_x \equiv -\ell^2 \qquad C_y \equiv -k^2 \longrightarrow C_z = k^2 + \ell^2$$

$$\boxed{\frac{\partial^2 Z}{\partial z^2} = +(k^2+\ell^2)\,Z} \qquad \boxed{\frac{\partial^2 X}{\partial x^2} = -\ell^2 X} \qquad \boxed{\frac{\partial^2 Y}{\partial y^2} = -k^2 Y}$$

One needs then to solve the three ordinary differential equations above

$$Z(z) = A\,e^{\sqrt{k^2+\ell^2}\,z} + B\,e^{-\sqrt{k^2+\ell^2}\,z}$$

$$Y(y) = C\sin ky + D\cos ky$$

$$X(x) = E\sin \ell x + F\cos \ell x$$

The boundary conditions imply that several of the constants A,...,F vanish:

$$5) \longrightarrow A = 0 \qquad 3) \longrightarrow D = 0$$

$$1) \longrightarrow F = 0$$

In addition

$$4) \longrightarrow k = \frac{n\pi}{a} \qquad 2) \longrightarrow p = \frac{m\pi}{b}$$

with $\quad n, m = 1, 2, 3, 4, \ldots$

Taking into account these facts, the factored solutions look like

$$\boxed{\varphi(x,y,z) = C \; \exp\left(-\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,z\right)\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}x\right)}$$

Starting from the factored function above one can build the most general solution for the Laplace equation in the pipe as a double sum over the indices n and m

$$\varphi(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \exp\left(-\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,z\right)\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}x\right)$$

Finally, one needs to account for the boundary condition 6) $\varphi(x,y,0) = V(x,y)$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}x\right) = V(x,y)$$

One now needs first to find an integral form for the constants C. One can use the orthogonality of the sin functions and find

$$\sum_{n,m} C_{n,m} \underbrace{\int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right)dy}_{\frac{a}{2}\delta_{nn'}} \times$$

$$\times \underbrace{\int_0^b \sin\left(\frac{m\pi}{b}x\right)\sin\left(\frac{m'\pi}{b}x\right)dx}_{\frac{b}{2}\delta_{mm'}} =$$

$$= \int_0^a \int_0^b dx\, dy\; V(x,y)\sin\left(\frac{n'\pi}{a}y\right)\sin\left(\frac{m'\pi}{b}x\right)$$

$$C_{n,m} = \frac{4}{ab}\int_0^a dy \int_0^b dx\; V(x,y)\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}x\right)$$

Consider now the special case in which the function V is a constant

$$C_{n,m} = \frac{4V}{ab}\int_0^a dy\,\sin\left(\frac{n\pi}{a}y\right)\int_0^b dx\,\sin\left(\frac{m\pi}{b}x\right)$$

$$= \frac{4V}{ab}\left[-\frac{a}{n\pi}\cos u\Big|_0^{n\pi}\right]\left[-\frac{b}{m\pi}\cos r\Big|_0^{m\pi}\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ or } m \text{ is even} \\ \dfrac{16V}{nm\pi^2} & \text{if both } n \text{ and } m \text{ are odd} \end{cases}$$

Finally then, for q constant V

$$\boxed{\varphi(x,y) = \frac{16V}{\pi^2}\sum_{n,m=odd}\frac{1}{nm}\exp\left(-\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,z\right)\times \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}x\right)}$$