Electrostatic total energy

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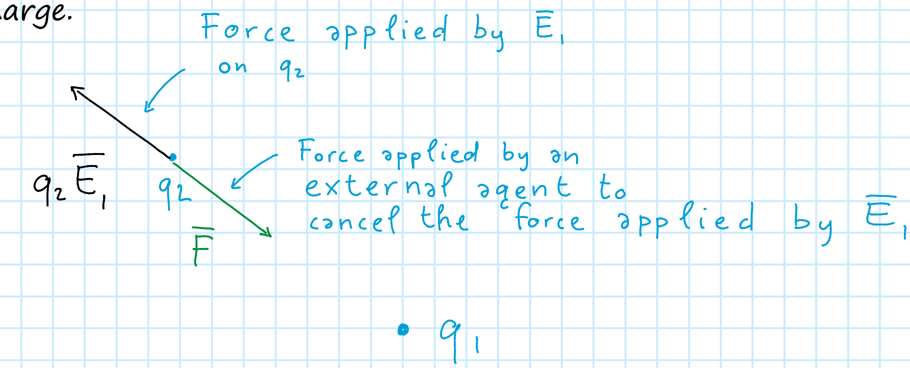
The total electrostatic energy of a system of charges is defined as the total work needed in order to assemble a give charge configuration starting from a state in which all of the charges are infinitely far from one another. The total electrostatic energy of the configuration in which all of the charges are infinitely distant from one another is defined to be zero.

Let's calculate this energy in the case of a set of point like charges. No work against the field E is done to bring the first charge in place, since there is initially no E in the region of space where we want to place the charges. In short, no work is done in bringing the first charge at the position r_1.

One can then put the second charge into place by bringing it in from infinity applying at each instant a force which cancels exactly the force applied on the second charge by the electric field created by the first charge.



$$W_{12} = \int_{\infty}^{\bar{r}_2} d\bar{\ell} \cdot \bar{F} = -q_2 \int_{\infty}^{\bar{r}_2} d\bar{\ell} \cdot \bar{E}_1(\bar{r}) = +q_2 \int_{\infty}^{\bar{r}_2} d\bar{\ell} \cdot \nabla \varphi_1(\bar{r}) = q_2 \left[\varphi_1(\bar{r}_2) - \cancelto{0}{\varphi_1(\infty)} \right]$$

This can also be written as

$$W_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\bar{r}_1 - \bar{r}_2|}$$

In order to add a third charge we have to add the corresponding contributions

$$W_{13} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_3}{|\bar{r}_1 - \bar{r}_3|} \qquad W_{23} = \frac{1}{4\pi\varepsilon_0} \frac{q_2 q_3}{|\bar{r}_2 - \bar{r}_3|}$$

So that

$$U = W_{TOT} = W_{12} + W_{13} + W_{23}$$

In general, for a configuration made of n point-like charges one will find

$$U = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n-1} \sum_{j>i}^{n} \frac{q_i q_j}{|\bar{r}_i - \bar{r}_j|} = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\neq j}}^{n} \frac{q_i q_j}{|\bar{r}_i - \bar{r}_j|}$$

$$= \frac{1}{2} \sum_{i=1}^{n} q_i \varphi(\bar{r}_i)$$

If one deals with a continuous charge distribution the total electrostatic energy will be

$$q \rightarrow \rho\, d^3r \qquad \qquad \Sigma \rightarrow \int$$

$$\boxed{U = \frac{1}{8\pi\varepsilon_0} \int d^3r \int d^3r' \, \frac{\rho(\bar{r})\rho(\bar{r}')}{|\bar{r} - \bar{r}'|} = \frac{1}{2} \int d^3r\, \rho(\bar{r})\, \varphi(\bar{r})}$$

U in terms of E

We now write U in terms of E and we show that U written in integral form is a positive quantity. (U written as a summation for a collection of point charges is not guaranteed to be positive, because in defining it we excluded the - infinite - self energy contributions. Point charges must be treated carefully!).

$$U = \frac{1}{2} \int_V d^3r\, \rho(\bar{r})\, \varphi(\bar{r}) = \frac{1}{2} \int_V d^3r \left(\varepsilon_0 \nabla \cdot \bar{E}\right) \varphi =$$

$$= \frac{\varepsilon_0}{2} \underbrace{\int d^3r\, \nabla \cdot (\bar{E}\varphi)} - \frac{\varepsilon_0}{2} \int_V d^3r\, \bar{E} \cdot \nabla\varphi = \frac{\varepsilon_0}{2} \int d^3r\, \bar{E}\cdot\bar{E} = \frac{\varepsilon_0}{2} \int d^3r\, |\bar{E}|^2 \geq 0$$

$$= \int_{\partial V} d\bar{s} \cdot \bar{E}\varphi = 0$$

Fields vanish sufficiently fast at ∞

The equation above suggests an electrostatic energy density of the form

$$u(\bar{r}) = \frac{\varepsilon_0}{2} |E(\bar{r})|^2$$