Electric and magnetic fields in matter

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It became clear to William Thompson (Lord Kelvin), that the magnetic field produced by Faraday induction was in some aspects different from the field induced by a steady current in matter (indicated by H). Similarly Maxwell figured out that the electric field responsible for the attraction or repulsion between charged objects was different from the electric field induced in matter by external charges (indicated by D). We now know why B and H are different and why E and D are different, but when dealing with fields in matter it is still useful to work with all of the four fields E,D,B and H.

Here we just define these fields and show Maxwell equations in matter. We will deal with their physical interpretation later.

First we state that the macroscopic charged density inside an isolated sample of neutral matter is zero in every point. The same applies to the current density in an isolated sample of matter (with the exception of ferromagnets):

$$\boxed{\rho(\bar{r}) = 0 \qquad ; \qquad \bar{J}(\bar{r}) = 0}$$

In presence of external charges and currents, the microscopic charges in a material (electrons, nuclei) rearrange themselves and their reciprocal positions, creating non zero charge and current densities in the material. To describe this it is traditional to define a polarization vector P and a magnetization vector M

$$\underbrace{\rho(\bar{r},t)}_{\substack{\text{total}\\\text{charge}\\\text{density}}} = \underbrace{\rho_f(\bar{r},t)}_{\substack{\text{"free"}\\\text{charge}\\\text{density}}} \underbrace{- \nabla\cdot\bar{P}(\bar{r},t)}_{\substack{\text{bound charge}\\\text{density } \rho_b}}$$

$$\underbrace{\bar{J}(\bar{r},t)}_{\substack{\text{total current}\\\text{density}}} = \underbrace{\bar{J}_f(\bar{r},t)}_{\substack{\text{free current}\\\text{density}}} + \nabla\times\bar{M}(\bar{r},t) + \frac{\partial\bar{P}(r,t)}{\partial t}$$

One can now introduce the macroscopic fields D and H as follows

$$\bar{D}(\bar{r},t) = \varepsilon_0 \bar{E}(\bar{r},t) + \bar{P}(\bar{r},t)$$

$$\bar{H}(\bar{r},t) = \frac{\bar{B}(\bar{r},t)}{\mu_0} - \bar{M}(\bar{r},t)$$

Now let's figure out what are the equations satisfied by these fields:

$$\nabla\cdot\bar{D} = \varepsilon_0 \underbrace{\nabla\cdot\bar{E}}_{=\rho/\varepsilon_0} + \nabla\cdot\bar{P} = \rho + \nabla\cdot\bar{P} = \rho_f$$

$$\nabla\times\bar{H} = \frac{1}{\mu_0}\underbrace{\nabla\times\bar{B}}_{=\frac{1}{c^2}\frac{\partial\bar{E}}{\partial t}+\mu_0\bar{J}} - \nabla\times\bar{M} = \frac{1}{\mu_0 c^2}\frac{\partial\bar{E}}{\partial t} + \bar{J} - \nabla\times\bar{M}$$

$$= \underbrace{\frac{1}{\mu_0 c^2}\frac{1}{\varepsilon_0}}_{=1}\frac{\partial\bar{D}}{\partial t} \overbrace{- \underbrace{\frac{1}{\mu_0 c^2}\frac{1}{\varepsilon_0}}_{1}\frac{\partial\bar{P}}{\partial t} + \bar{J} - \nabla\times\bar{M}}^{\text{this is } \bar{J}_f}$$

$$= \frac{\partial\bar{D}}{\partial t} + \bar{J}_f$$

From these equations, using the same techniques already employed for E and B, one finds the following boundary conditions for D and H (same notation as for E,B)

$$\hat{n}\cdot\left[\bar{D}_1 - \bar{D}_2\right] = \sigma_f \quad , \quad \hat{n}\times\left[\bar{H}_1 - \bar{H}_2\right] = \bar{K}_f$$

We have now a total of six equations, but we need to fix four vector fields (E,B,D,H) each one of which has three components. The four equations below are not sufficient for this task:

ME in matter

$$\nabla\cdot\bar{B} = 0 \quad , \quad \nabla\times\bar{E} + \frac{\partial\bar{B}}{\partial t} = 0 \; ;$$

$$\nabla \cdot \overline{D} = \rho_f \quad , \qquad \nabla \times \overline{H} - \frac{\partial \overline{D}}{\partial t} = \mu_0 \overline{J}$$

We need six additional conditions that link the components of H and D to the components of B and E

$$\overline{D} = \overline{D}(\overline{E}, \overline{B}) \quad ; \qquad \overline{H} = \overline{H}(\overline{E}, \overline{B})$$

These relations are called constitutive relations. When the fields are not strong, these relations are linear. We often work in the limit of static fields; in that case we call "simple media" materials that satisfy the relations

$$\overline{D}(\overline{r}) = \underbrace{\varepsilon}_{\text{dielectric permittivity}} \overline{E}(r) \qquad \overline{H}(\overline{r}) = \frac{\overline{B}(r)}{\underbrace{\mu}_{\text{magnetic permeability}}}$$

$$\overline{P}(\overline{r}) = \varepsilon_0 \underbrace{\chi}_{\text{electric susceptibility}} \overline{E}(\overline{r}) \qquad \overline{M}(\overline{r}) = \underbrace{\chi_m}_{\text{magnetic susceptibility}} \overline{H}(\overline{r})$$

Consequently,

$$\overline{D} = \varepsilon_0 \overline{E} + \overbrace{\varepsilon_0 \chi \overline{E}}^{\overline{P}} = \varepsilon_0 (1 + \chi) \overline{E}$$

$$\boxed{\varepsilon = \varepsilon_0 (1 + \chi)}$$

$$\overline{H} = \frac{1}{\mu_0} \overline{B} - \overline{M} = \frac{1}{\mu_0} \overline{B} - \chi_m \overline{H}$$

$$(1 + \chi_m) \overline{H} = \frac{1}{\mu_0} \overline{B}$$

$$\overline{H} = \frac{\overline{B}}{\mu_0 (1 + \chi_m)}$$

$$\boxed{\mu = \mu_0 (1 + \chi_m)}$$