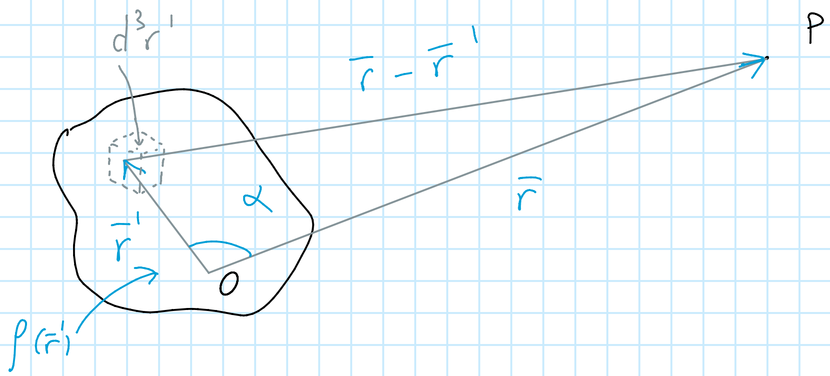
Multipole expansion

Monday, December 31, 2018 3:53 AM

If one is looking to a generic charge distribution from far away, is it possible to systematically calculate the potential in a series expansion in which subsequent terms become numerically less relevant?



Let's restart from the generic expression for the potential

$$\varphi(\bar r) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(\bar r')}{|\bar r - \bar r'|}$$

Our goal is now to expand the denominator in the integral under the assumption that r >> r'

$$\tau \equiv |\bar r - \bar r'|$$

$$\tau^2 = r^2 + r'^2 - 2rr'\cos\alpha = r^2\Big[1 + \underbrace{\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha}_{\equiv\, \epsilon \,<<\, 1}\Big]$$

$$\hookrightarrow \quad \tau = r\sqrt{1+\epsilon}$$

$$\frac{1}{\tau} = \frac{1}{|\bar r - \bar r'|} = \frac{1}{r}\,\frac{1}{\sqrt{1+\epsilon}} = \frac{1}{r}\left(1 - \frac{\epsilon}{2} + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots\right)$$

$$= \frac{1}{r}\Big[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2$$

$$- \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \dots\Big]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \right.$$

$$\left. + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \dots\right]$$

The polynomials depending on $\cos\alpha$ are a special set of polynomials called Legendre Polynomials. They satisfy several interesting relations and they appear in this expansion since one can in general write

$$\frac{1}{\tau} = \frac{1}{|\bar{r} - \bar{r}'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \underbrace{P_n(\cos\alpha)}_{\text{Legendre polynomial}}$$

$$P_0(\cos\alpha) = 1 \qquad P_1(\cos\alpha) = \cos\alpha$$

$$P_2(\cos\alpha) = \frac{1}{2}\left(3\cos^2\alpha - 1\right)$$

$$P_3(\cos\alpha) = \frac{1}{2}\left(5\cos^3\alpha - 3\cos\alpha\right) \quad \text{etc.}$$

The potential can then be written as

$$\boxed{\varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\,\rho(\bar{r}')\,d^3r'}$$

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$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\underbrace{\int \rho(\bar{r}')\,d^3r'}_{Q_0} + \frac{1}{r^2}\underbrace{\int r'\cos\alpha\,\rho(\bar{r}')\,d^3r'}_{Q_1}\right.$$

$$\left. + \frac{1}{r^3}\underbrace{\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\bar{r}')\,d^3r'}_{Q_2} + \dots\right]$$

$Q_0 \longrightarrow$ monopole term

$Q_1 \longrightarrow$ dipole term

$Q_2 \longrightarrow$ quadrupole term

Example 1

(from Griffith, problem 27)

Consider a sphere of radius R, centered at the origin of the reference frame, which has the following charge density

$$\rho(\bar{r}) = k \frac{R}{r^2} (R - 2r) \sin\vartheta$$

Find the potential for a point along the z axis

$$Q_0 = \int d^3r' \, \rho(r')$$

$$= \int_0^R dr \, r^2 \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\phi \; k \frac{R}{r^2} (R - 2r) \sin\vartheta$$

consider the integral in r

$$\int_0^R dr \, (R - 2r) = Rr - r^2 \Big|_0^R = 0$$

$Q_0 = 0$

no monopole

$$Q_1 = \int d^3r' \; r' \cos\alpha \; \rho(\bar{r}')$$

since we look at a point along the z-axis, $\alpha = \vartheta$

$$= \int_0^R dr' (r')^2 \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\phi \; \cos\vartheta \; \kappa \frac{R}{(r')^2} \left(R - 2r'\right) \sin\vartheta$$

consider the integral in ϑ

$$\int_{-1}^{1} d\cos\vartheta \; \cos\vartheta \sin\vartheta = \underbrace{\int_{-1}^{1} dx}_{\text{symm interval}} \; \underbrace{x\sqrt{1-x^2}}_{\text{odd function}} = 0$$

$Q_1 = 0$ no dipole

$$Q_2 = \int d^3r' (r')^2 \; P_2(\cos\alpha) \; \rho(\bar{r}')$$

$$= \int_0^R dr' (r')^2 \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\phi \cancel{(r')^2} \kappa \frac{R}{\cancel{(r')^2}} \left(R - 2r'\right) \sin\vartheta$$

$$\times \frac{3}{2}\left(\cos^2\vartheta - 1\right) = *$$

$$\int_0^R dr \; r^2 (R - 2r) = R\frac{r^3}{3} - 2\frac{r^4}{4}\Big|_0^R = R^4\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= -\frac{R^4}{6}$$

$$\int_{-1}^{1} dx \left(\frac{3}{2}x^2 - \frac{1}{2}\right)\sqrt{1-x^2} = \frac{3}{2}\frac{\pi}{8} - \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{2}\left(\frac{3}{8} - \frac{1}{2}\right)$$

$$= -\frac{\pi}{16}$$

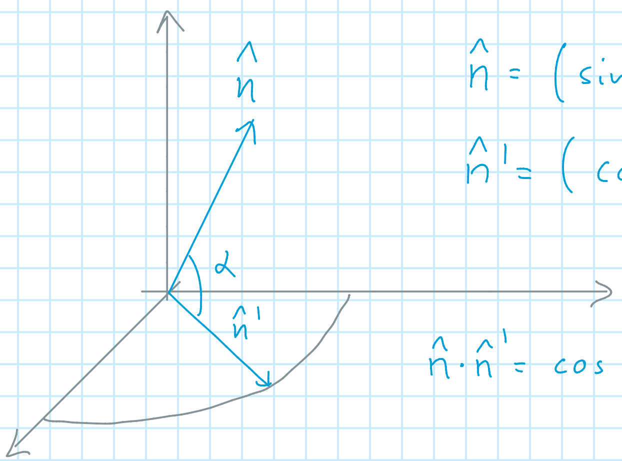
$$Q_2 = * = \kappa R \, 2\pi \frac{R^4}{6} \frac{\pi}{16} = \frac{\kappa \pi^2}{48} R^5$$

$$\varphi = \frac{1}{4\pi\varepsilon_0} \frac{\kappa\pi^2}{48} \frac{R^5}{r^3} + \dots$$

Example 2 (from Griffith problem 28)

A circular ring in the x-y plane has radius R and it is centered at the origin. The ring carries a constant charge per unit length lambda. Evaluate the first three terms in the Multipole expansion for an arbitrary point of spherical coordinates r and theta.

$$Q_0 = \int d\ell \, \lambda = R \int d\phi \, \lambda = 2\pi R \lambda$$



$$\hat{n} = (\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta)$$

$$\hat{n}' = (\cos\phi', \sin\phi', 0)$$

$$\hat{n}\cdot\hat{n}' = \cos\alpha = \sin\vartheta(\cos\phi\cos\phi' + \sin\phi\sin\phi')$$

$$= \sin\vartheta\cos(\phi-\phi')$$

$$Q_1 = \int d\ell \, r' \cos\alpha \, \lambda$$

$$= R^2 \lambda \sin\vartheta \int_0^{2\pi} d\phi' \cos(\phi-\phi') = 0$$

$$Q_2 = \int d\ell \, (r')^2 \, \frac{1}{2}\left(3\cos^2\alpha - 1\right) \lambda$$

$$= \frac{R^3\lambda}{2} \int_0^{2\pi} d\phi' \left[3\sin^2\vartheta \cos^2(\phi - \phi') - 1\right]$$

$$= \frac{R^3\lambda}{2} \left[3\sin^2\vartheta \int_0^{2\pi} d\phi' \cos^2(\phi - \phi') - \int_0^{2\pi} d\phi'\right]$$

$$= \frac{R^3\lambda}{2}\left[3\sin^2\vartheta \, \pi - 2\pi\right] = \frac{R^3\lambda\pi}{2}\left(3\sin^2\vartheta - 2\right)$$