Method of images - dipoles

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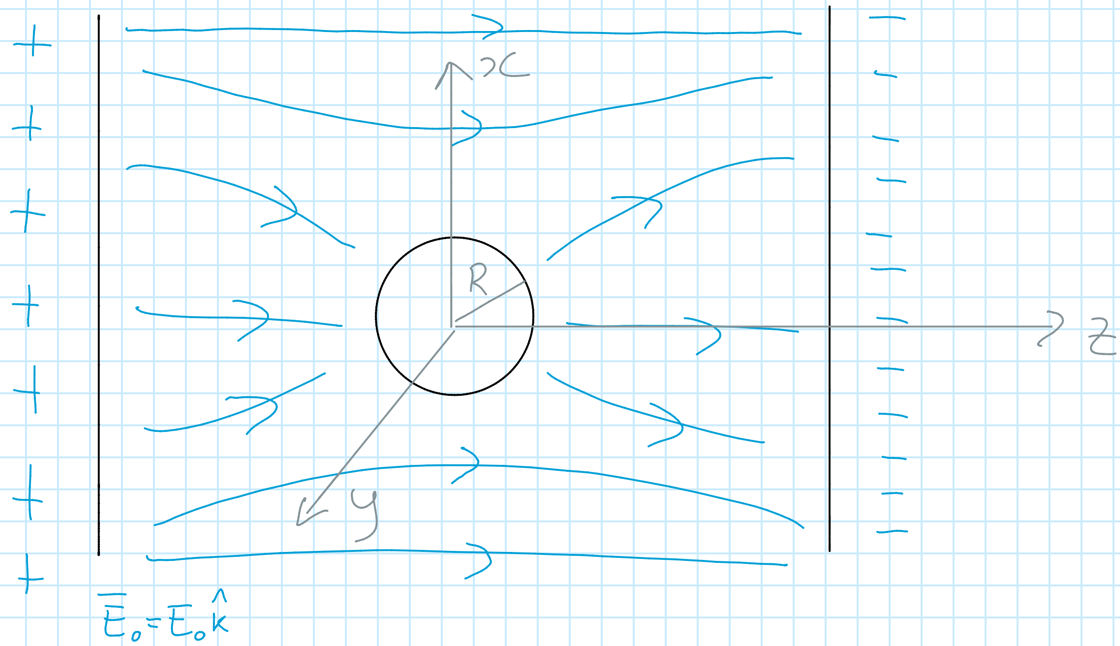
One can use an "Image dipole" to find the potential in the space surrounding a conducting, neutral sphere placed in a field that was initially constants. This situation can be visualized by imagining to place the sphere in the space between two infinitely large, very distant capacitor plates.



We can add an image dipole (in the space occupied by the sphere) to the potential responsible for the contact field. This combination solves the Laplace equation because each one of these two potential solves separately the Laplace equation.

$$\varphi = -E_0 r \cos\vartheta + \frac{p}{4\pi\varepsilon_0}\frac{\cos\vartheta}{r^2}$$

Now we want the potential to be constant on the surface of the conductor. Let's choose this constant to be zero

$$\varphi(r=R) = 0 = -E_0 R \cos\vartheta + \frac{p}{4\pi\varepsilon_0}\frac{\cos\vartheta}{R^2}$$

$$\hookrightarrow \quad p = 4\pi\varepsilon_0 E_0 R^3$$

$$\varphi = -E_0 \cos\vartheta \left(r - \frac{R^3}{r^2}\right)$$

One can then calculate the surface charge density on the surface of the sphere by observing that the electric field just outside the sphere will be

$$E_r = -\frac{\partial}{\partial r}\varphi = E_0 \cos\vartheta \left(1 + \frac{2R^3}{r^3}\right)$$

$$\sigma = \varepsilon_0 E_r \Big|_{r=R} = 3\varepsilon_0 E_0 \cos\vartheta$$

The total charge on the sphere can be obtained by integrating the surface charge over the sphere

$$Q = \int ds \ \sigma = R^2 \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\vartheta \, 3\varepsilon_0 E_0 \cos\vartheta = 0$$

$$\text{since} \quad \int_{-1}^{1} dx \ x = \frac{x^2}{2}\Big|_{-1}^{1} = 0$$

The total charge is zero as expected, since the sphere is isolated.