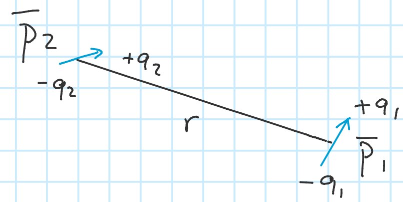
Force between dipoles

Wednesday, February 20, 2019 6:15 PM

We already evaluated what is the potential due to a dipole. We now want to evaluate the electric potential energy of a second dipole placed at some point in the space surrounding the first dipole.



$$\bar{p}_2 \equiv q_2 \vec{d}_2 \qquad \bar{p}_1 \equiv q_1 \vec{d}_1 \qquad r \gg d_1, d_2$$

$$\varphi_1(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\bar{p}_1 \cdot \bar{r}}{r^3}$$

$$U = \frac{1}{2}\sum_i q_i \varphi(\bar{r}_i) = \frac{q_2}{2}\varphi\left(\bar{r}+\frac{\bar{d}_2}{2}\right) - \frac{q_2}{2}\varphi\left(\bar{r}-\frac{\bar{d}_2}{2}\right)$$

$$= \frac{q_2}{8\pi\varepsilon_0}\left(\frac{\bar{p}_1\cdot\left(\bar{r}+\frac{\bar{d}_2}{2}\right)}{\left|\bar{r}+\frac{\bar{d}_2}{2}\right|^3} - \frac{\bar{p}_1\cdot\left(\bar{r}-\frac{\bar{d}_2}{2}\right)}{\left|\bar{r}-\frac{\bar{d}_2}{2}\right|^3}\right)$$

$$= \frac{q_2}{8\pi\varepsilon_0}\left[\frac{\bar{p}_1\cdot\left(\bar{r}+\frac{\bar{d}_2}{2}\right)}{r^3}\left(1+\frac{\bar{r}\cdot\bar{d}_2}{r^2}+\dots\right)^{-\frac{3}{2}}\right.$$

$$\left. - \frac{\bar{p}_1\cdot\left(\bar{r}-\frac{\bar{d}_2}{2}\right)}{r^3}\left(1-\frac{\bar{r}\cdot\bar{d}_2}{r^2}+\dots\right)^{-\frac{3}{2}}\right]$$

$$= \frac{q_2}{8\pi\varepsilon_0}\left[\frac{\bar{p}_1\cdot\left(\bar{r}+\frac{\bar{d}_2}{2}\right)}{r^3}\left(1-\frac{3}{2}\frac{\bar{r}\cdot\bar{d}_2}{r^2}+\dots\right)\right.$$

$$\left. - \frac{\bar{p}_1\cdot\left(\bar{r}-\frac{\bar{d}_2}{2}\right)}{r^3}\left(1+\frac{3}{2}\frac{\bar{r}\cdot\bar{d}_2}{r^2}+\dots\right)\right]$$

$$= \frac{q_2}{8\pi\varepsilon_0}\left[\cancel{\frac{\bar{p}_1\cdot\bar{r}}{r^3}} + \frac{\bar{p}_1\cdot\bar{d}_2}{2r^3} - \frac{3}{2}\frac{\bar{p}_1\cdot\bar{r}\;\bar{d}_2\cdot\bar{r}}{r^5}\right.$$

$$\left. - \cancel{\frac{\bar{p}_1\cdot\bar{r}}{r^3}} + \frac{\bar{p}_1\cdot\bar{d}_2}{2r^3} - \frac{3}{2}\frac{\bar{p}_1\cdot\bar{r}\;\bar{d}_2\cdot\bar{r}}{r^5} + \dots\right]$$

$$= \frac{q_2}{8\pi\varepsilon_0}\left[\frac{\bar p_1\cdot \bar d_2}{r^3} - 3\,\frac{\bar p_1\cdot\bar r\;\; \bar d_2\cdot \bar r}{r^5}\right]$$

$$= \frac{1}{8\pi\varepsilon_0}\left[\frac{\bar p_1\cdot\bar p_2}{r^3} - \frac{3(\bar p_1\cdot\bar r)(\bar p_2\cdot\bar r)}{r^5}\right]$$

symmetric in $\bar p_1 \leftrightarrow \bar p_2$

The force applied by the first dipole on the second one is

$$\bar F = -\nabla U = \frac{1}{8\pi\varepsilon_0}\nabla\left(\frac{3(\bar p_1\cdot\bar r)(\bar p_2\cdot\bar r)}{r^5} - \frac{\bar p_1\cdot\bar p_2}{r^3}\right)$$

$$\nabla\;\bar p\cdot\bar r = \left(\frac{\partial}{\partial x_j}\;p_i x_i\right)\hat\imath_j = \bar p$$

$$\hookrightarrow\ \bar F = \frac{1}{8\pi\varepsilon_0}\left\{\frac{3\left[\nabla(\bar p_1\cdot\bar r)\right](\bar p_2\cdot\bar r)}{r^5} + \frac{3(\bar p_1\cdot\bar r)\left[\nabla(\bar p_2\cdot\bar r)\right]}{r^5}\right.$$

$$\left. + 3(\bar p_1\cdot\bar r)(\bar p_2\cdot\bar r)\,\nabla\frac{1}{r^5} - \bar p_1\cdot\bar p_2\,\nabla\frac{1}{r^3}\right\}$$

$$= \frac{1}{8\pi\varepsilon_0}\left\{\frac{3\,\bar p_1(\bar p_2\cdot\bar r) + 3(\bar p_1\cdot\bar r)\,\bar p_2}{r^5}\right.$$

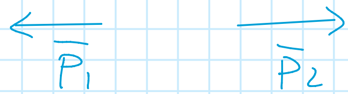
$$\left. - 15(\bar p_1\cdot\bar r)(\bar p_2\cdot\bar r)\frac{\hat r}{r^6} + 3\,\frac{\bar p_1\cdot\bar p_2}{r^4}\,\hat r\right\}$$

$$\boxed{= \frac{3}{8\pi\varepsilon_0}\frac{1}{r^4}\left[\bar p_1(\bar p_2\cdot\hat r) + \bar p_2(\bar p_1\cdot\hat r) + \bar p_1\cdot\bar p_2\,\hat r - 5(\bar p_1\cdot\hat r)(\bar p_2\cdot\hat r)\hat r\right]}$$

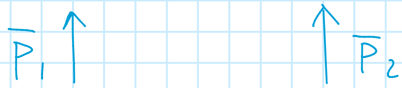
Special cases

$\xrightarrow{\bar p_1}$ $\quad\xrightarrow{\hat r}\quad$ $\xrightarrow{\bar p_2}$

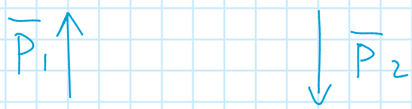
$$\bar F = \frac{3}{8\pi\varepsilon_0}\hat r\frac{1}{r^4}\left[-2p_1p_2\right]$$ attractive



$$\bar{F} = \frac{3\hat{r}}{8\pi\varepsilon_0 r^4}\left[2p_1p_2\right]$$ repulsive



$$\bar{F} = \frac{3}{8\pi\varepsilon_0}\frac{\hat{r}}{r^4}\left[p_1p_2\right]$$ repulsive



$$\bar{F} = \frac{3}{8\pi\varepsilon_0}\frac{\hat{r}}{r^4}\left[-p_1p_2\right]$$ attractive