Point charge in presence of a charged insulated conducting sphere.

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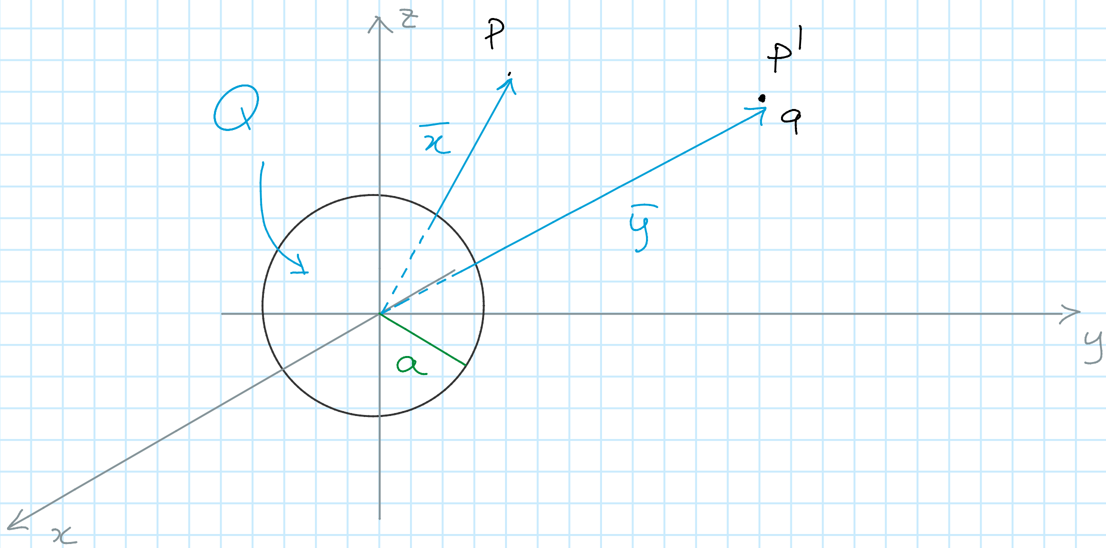

One can solve this problem by using the solution already found for the case of a grounded sphere. One can imagine to take the grounded sphere, on which we already have the induced charge q' determined in the previous set of notes, disconnect it from the ground and add a charge $Q - q'$, where Q indicates the total charge that one wants to have on a conducting sphere. The forces due to the point like charge q are already balanced by the charge q' discussed in the case of the grounded sphere. Consequently, the additional charge $Q - q'$ will distribute itself uniformly over the sphere. For points outside the sphere, $Q-q'$ will produce the same effect as a point charge of that magnitude placed at the center of the sphere.

The potential for this case is then

$$\varphi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\bar{x} - \bar{y}|} - \frac{q\,a}{y\left|\bar{x} - \frac{a^2}{y^2}\bar{y}\right|} + \frac{Q + \frac{a}{y} q}{|\bar{x}|} \right)$$

$$\left(\text{rem} \quad y' = \frac{a^2}{y} \qquad q' = -\frac{q}{y} a \right)$$

Then, as expected, the potential for a point on the sphere is simply

$$\bar{x} = a\hat{n}$$

$$\varphi(|\bar{x}| = a) = \frac{1}{4\pi\varepsilon_0}\,\frac{Q + \frac{a}{y}q}{a}$$

The force applied by the sphere on the point charge q will be

$$\bar{F} = \frac{1}{4\pi\varepsilon_0}\left(\frac{qq'}{(y-y')^2} + \frac{(Q-q')q}{y^2}\right)\hat{n}'$$

$$= \frac{1}{4\pi\varepsilon_0}\left[-\frac{q^2 a}{y\left(y - \frac{a^2}{y}\right)^2} + \frac{Qq + \frac{q^2 a}{y}}{y^2}\right]\hat{n}'$$

$$[\dots] = \frac{q}{y^2}\left(-\frac{qay}{\left(y - \frac{a^2}{y}\right)^2} + Q + \frac{qa}{y}\right)$$

$$= \frac{q}{y^2}\left(-\frac{qay^3}{(y^2-a^2)^2} + Q + \frac{qa}{y}\right)$$

$$= \frac{q}{y^2}\left[Q - qa\left(\frac{y^3}{(y^2-a^2)^2} - \frac{1}{y}\right)\right]$$

$$= \frac{q}{y^2}\left[Q - qa\left(\frac{\cancel{y^4} - \cancel{y^4} - a^4 + 2a^2y^2}{y(y^2-a^2)^2}\right)\right]$$

$$= \frac{q}{y^2}\left[Q - qa\,\frac{2a^2y^2 - a^4}{y(y^2-a^2)^2}\right] = \frac{q}{y^2}\left[Q - qa^3\,\frac{2y^2 - a^2}{y(y^2-a^2)}\right]$$

$$\bar{F} = \frac{1}{4\pi\varepsilon_0} \frac{q}{y^2}\left[Q - \frac{q a^3 (2y^2 - a^2)}{y(y^2-a^2)^2} \right] \frac{\bar{y}}{y}$$

$\hat{n}'$

For $y >> a$ one finds, as expected,

$$\bar{F} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{y^2} \frac{\bar{y}}{y}$$

When Q and q have opposite signs, or when Q = 0, the force is attractive for all values of y.

However, even if Q and q have the same sign, the force is attractive when the second term in the equation for the force dominates. One can study this phenomenon by assuming that $0 < q/Q << 1$ and subsequently looking for the value of y at which the first (repulsive) term in the force equation becomes as big as the second term (which is attractive)

$$Q - q \frac{a^3(2y^2 - a^2)}{y(y^2-a^2)^2} = 0$$

$$Q - q \frac{a^5\left(2\frac{y^2}{a^2} - 1\right)}{y a^4 \left(\frac{y^2}{a^2} - 1\right)^2} = Q - q \frac{2\frac{y^2}{a^2} - 1}{\frac{y}{a}\left(\frac{y^2}{a^2} - 1\right)^2} = 0$$

set $\quad z \equiv \frac{y}{a} \qquad r \equiv \frac{q}{Q}$

$$1 - r \frac{2z^2 - 1}{z(z^2-1)^2} = 0$$

$$z(z^2-1)^2 = r(2z^2-1)$$

If r = 0 the solution is z = 1. $(\text{rem } y \geq a, \; z \geq 1)$

One can then look at the first order corrections to the result z = 1

$$z \equiv 1 + \delta$$

$$\underbrace{(1+\delta)}_{z}\,\underbrace{(\cancel{1}+\delta^2+2\delta-\cancel{1})}_{z^2-1}{}^2 = r\,(\underbrace{2+2\delta^2+4\delta-1}_{2z^2-1})$$

$$4\delta^2 = r\,(1-4\delta) \;\rightarrow\; r = 4\delta^2 + \mathcal{O}(\delta^3)$$

$$\hookrightarrow \quad \delta = \frac{1}{2}\sqrt{r} = \frac{1}{2}\sqrt{\frac{q}{Q}}$$

$$\hookrightarrow \quad \boxed{y = a\left(1 + \frac{1}{2}\sqrt{\frac{q}{Q}}\right)}$$

value of y for where repulsive = attractive

The presence of the attractive term in the equation for the force explains why when there is an excess of charge of a given sign on the surface of the conductor the charges are not simply pushed out of the conductor. When a charge leaves the surface of the conductor it induces an image charge such that the force between the sphere and the charge that left the conductor is attractive (as long as the expelled charged is very close to the conductor's surface). It is only when work is done to bring the expelled charge at a certain distance from the conductor that the force between the conductor and the charge becomes repulsive (i.e. the first term in the force equation becomes larger than the second term).