Boundary Value Problems

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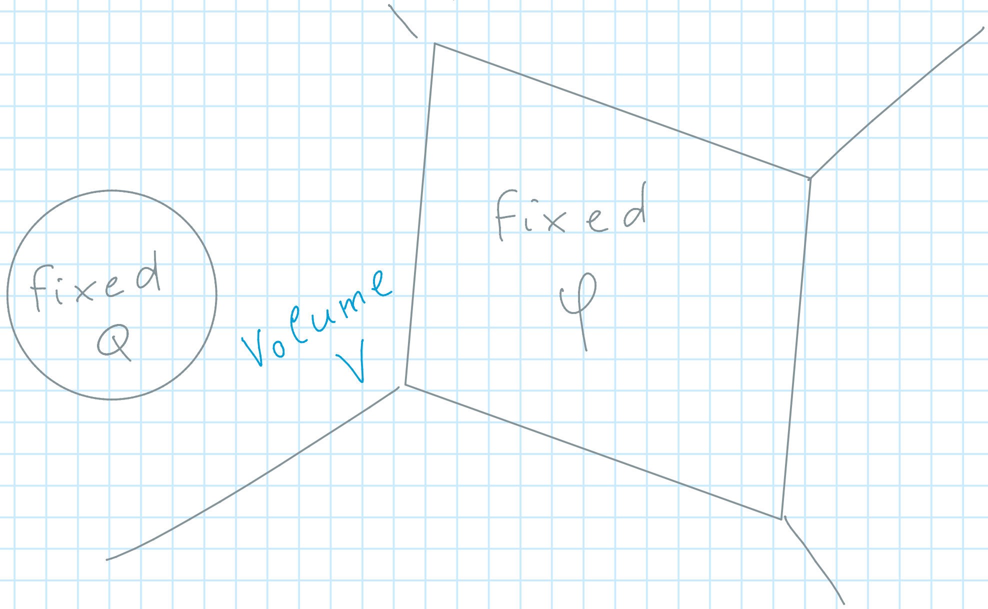
The surface of a conductor can be kept at constant potential by connecting it to a battery. This implies that charge can flow in or out of the conductor. A simple example of this situation is the case in which one connects a conductor to the surface of the Earth, which we take to have an electric potential of zero. In this case we say that the object is grounded.

One can then have to deal with conductors with a fixed charge or conductors with a fixed potential. If we have several conductors in the problem, the charges in the conductors will rearrange themselves in such a way that the electric fields in the conductors are zero and just outside the conductor they re perpendicular to the conductor's surface. In particular, if the surface charge density is σ

$$\boxed{\bar{E} = \frac{\sigma}{\varepsilon_0}\,\hat{n}}$$

$\hat{n}$ = unit vector $\perp$ to the conductor's surface, pointing out of the conductor's volume

In a boundary value problem, the goal is to determine the potential (and, consequently, the electric field) in the space between the conductors (or any other object with fixed charge or potential).



Since there are no charges in the space between the conductors, in the volume V which we are interested in the potential will satisfy Laplace's equation.

$$\nabla \cdot \bar{E} = 0 \quad \Rightarrow \quad \nabla \cdot \nabla \varphi = 0 \quad \Longrightarrow \quad \boxed{\Delta \varphi = 0}$$

In these problems, one can have two types of boundary conditions:
Dirichlet Boundary Conditions and Neumann Boundary Conditions

Dirichlet Boundary Conditions: The value of the potential is fixed on a given surface.
Neumann Boundary Conditions: The value of the electric field outside the surface of a conductor is fixed.

For any surface on the boundary of the volume which we are interested in we can choose either Dirichlet or Neumann boundary conditions, but not both.

Theorem If one chooses Dirichlet or Neumann boundary conditions on all of the surfaces which enclose a volume, the Laplace equation has a unique solution in that volume.

Proof Let's assume that there are two solutions to the Laplace equation, and let's indicate them with φ_1 and φ_2
One can then define

$$f = \varphi_1 - \varphi_2$$

Subsequently, we can integrate a particular function of f over the volume V

$$\int_V d^3x \; \nabla \cdot (f \nabla f) = \int_V d^3x \left\{ f \underbrace{\nabla \cdot \nabla f}_{\Delta f = 0} + (\nabla f) \cdot (\nabla f) \right\}$$

$$= \int_V d^3x \; (\nabla f) \cdot (\nabla f)$$

However, by applying the divergence theorem, one can rewrite the original integral as follows

$$\int_V d^3x \ \nabla \cdot (f \nabla f) = \int_S d\bar{s} \cdot f \nabla f = \sum_{i=1}^{n} \int_{S_i} d\bar{s} \cdot f \nabla f$$

On surfaces with Dirichlet boundary conditions, the integrals to the right vanish because f = 0 by definition. On surfaces with Neumann boundary conditions, the integrals vanish because the gradient of f is zero by definition. Therefore

$$\int_V d^3x \ \nabla \cdot \left(f \nabla f\right) = \int_V d^3x \ \nabla f \cdot \nabla f = \sum_{ii} \int_{S_i} d\bar{s} \cdot f \nabla f = 0$$

$$\int_V d^3x \left(\nabla f\right)^2 = 0$$

The gradient of f must be zero in all of the volume. If f is zero somewhere (Diruchlet boundary conditions) it must be zero everywhere. If there are only Neumann boundary conditions, f can be at most a constant. However two potentials that differ only by a constant everywhere describe the same electric field, which is therefore uniquely determined by the boundary conditions and the Laplace equation.