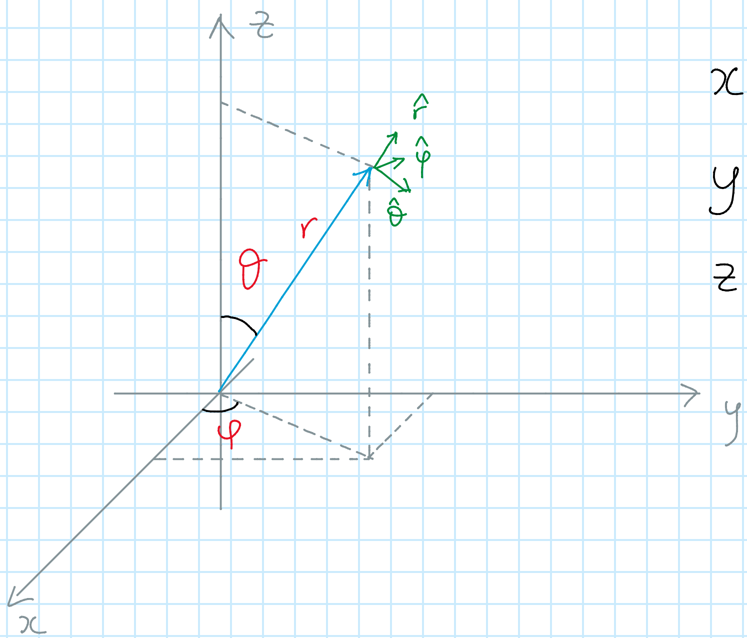
Curvilinear coordinates

Saturday, January 5, 2019 3:30 AM

Spherical coordinates

$$\{ r, \vartheta, \varphi \}$$



$$x = r \sin\vartheta \cos\varphi$$
$$y = r \sin\vartheta \sin\varphi$$
$$z = r \cos\vartheta$$

A completely generic vector can then be written as

$$\bar{A} = A_r \hat{r} + A_\vartheta \hat{\vartheta} + A_\varphi \hat{\varphi}$$

(however notice that a vector joining the origin of the frame of reference with a point P is by definition simply $\bar{r} = r \hat{r}$.)

The unit vectors change directions according to the space point that we consider.

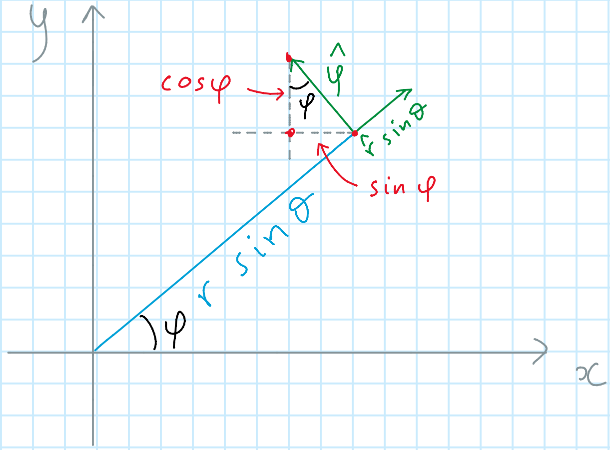
It is important to rewrite the unit vectors $\hat{r}, \hat{\vartheta}, \hat{\varphi}$ as a function of the unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$

$$\hat{r} = \sin\vartheta \cos\varphi \, \hat{\imath} + \sin\vartheta \sin\varphi \, \hat{\jmath} + \cos\vartheta \, \hat{k}$$
$$\hat{\vartheta} = \cos\vartheta \cos\varphi \, \hat{\imath} + \cos\vartheta \sin\varphi \, \hat{\jmath} - \sin\vartheta \, \hat{k}$$

replace $\vartheta \to \vartheta + \frac{\pi}{2}$ in $\hat{r}$

$$\hat{\varphi} = -\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}$$

The last relation can be found by looking at the x-y plane from above



Infinitesimal line element

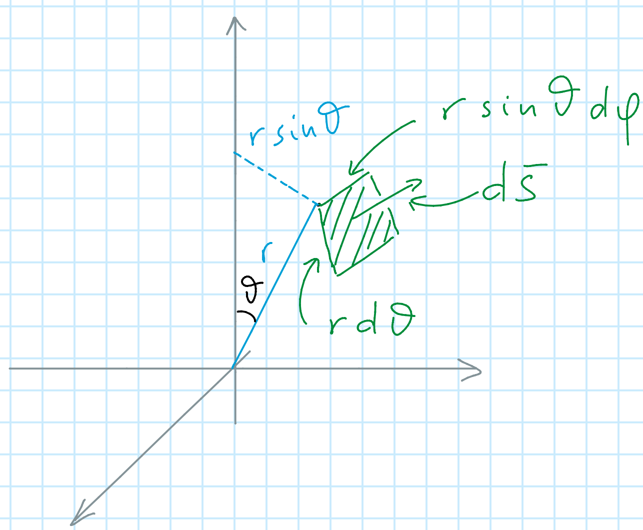
$$d\bar{\ell} = dr\,\hat{r} + r\,d\vartheta\,\hat{\vartheta} + r\sin\vartheta\,d\varphi\,\hat{\varphi}$$

Infinitesimal volume element

$$d^3x = dr\,d\vartheta\,d\varphi\,r^2\sin\vartheta = dr\,d\cos\vartheta\,d\varphi\,r^2$$

Infinitesimal surface element on a sphere of radius r

$$d\bar{S} = r^2\sin\vartheta\,d\vartheta\,d\varphi\,\hat{r}$$



Observe that

$$\int_{sphere} \hat{r} \cdot d\bar{s} = r^2 \int_0^{\pi} d\vartheta \sin\vartheta \int_0^{2\pi} d\varphi = 2\pi r^2 \int_{-1}^{1} dx = \underbrace{4\pi r^2}_{\text{AREA OF THE SPHERE}}$$

Gradient divergence and curl

It is important to be able to write the gradient, divergence and curl in spherical coordinates. The brute force way of doing this is to apply the chain rule

$$\nabla f = \partial_x f \,\hat{i} + \partial_y f \,\hat{j} + \partial_z f \,\hat{k}$$

$$\partial_x f \equiv \frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \vartheta} \frac{\partial \vartheta}{\partial x} + \frac{\partial f}{\partial \varphi} \frac{\partial \varphi}{\partial x}$$

and similar expressions for ∂_y and ∂_z

In order to calculate the derivative above, one needs the relations

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \sin\vartheta = \sqrt{\frac{x^2 + y^2}{x^2 + y^2 + z^2}}$$

$$\cos\varphi = \frac{x}{\sqrt{x^2 + y^2}}$$

This calculation is long, try to do it with a Computer Algebra System. Here we simply quote the result

$$\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \vartheta} \hat{\vartheta} + \frac{1}{r \sin\vartheta} \frac{\partial f}{\partial \varphi} \hat{\varphi}$$

$$\nabla \cdot \bar{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 v_r \right) + \frac{1}{r \sin\vartheta} \frac{\partial}{\partial \vartheta} \left(\sin\vartheta \, v_\vartheta \right) + \frac{1}{r \sin\vartheta} \frac{\partial v_\varphi}{\partial \varphi}$$

$$\nabla \times \bar{v} = \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(\sin\vartheta\, v_\varphi) - \frac{\partial v_\vartheta}{\partial\varphi}\right]\hat{r}$$

$$+\frac{1}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial v_r}{\partial\varphi} - \frac{\partial}{\partial r}(r\, v_\varphi)\right]\hat{\vartheta}$$

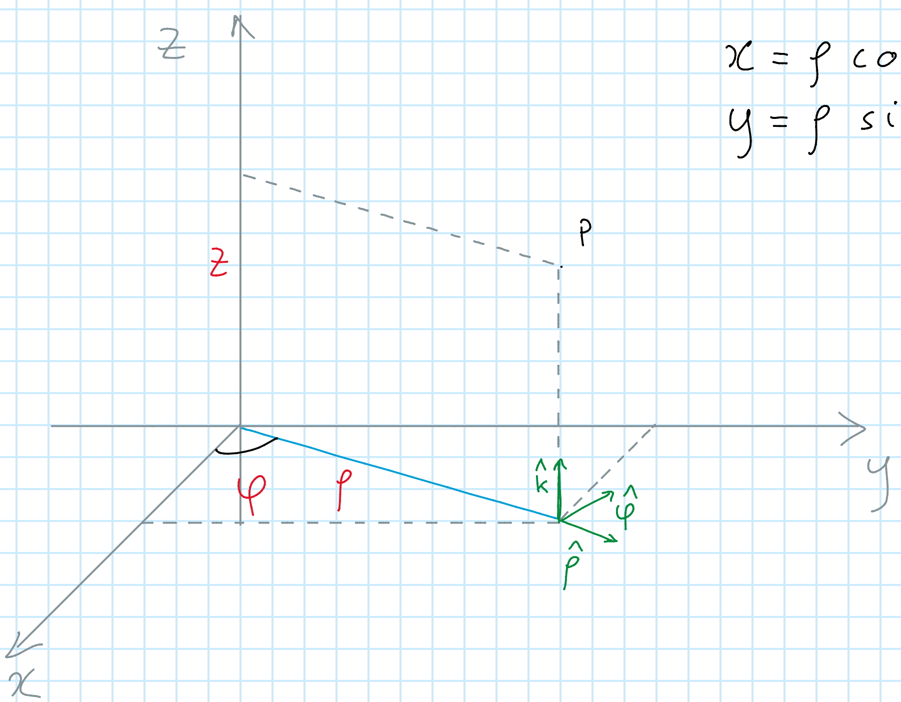
$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(r\, v_\vartheta) - \frac{\partial v_r}{\partial\varphi}\right]\hat{\varphi}$$

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 f}{\partial\varphi^2}$$

Cylindrical coordinates

$$\{\rho, \varphi, z\}$$

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$



$$\bar{A} = A_\rho \hat{\rho} + A_\varphi \hat{\varphi} + A_z \hat{k}$$

$$\hat{\rho} = \cos\varphi\, \hat{i} + \sin\varphi\, \hat{j}$$

$$\hat{\varphi} = -\sin\varphi\, \hat{i} + \cos\varphi\, \hat{j}$$

Infinitesimal line element

$$d\bar{\ell} = d\rho\, \hat{\rho} + \rho\, d\varphi\, \hat{\varphi} + dz\, \hat{k}$$

Infinitesimal volume element

$$d^3x = \rho\, d\rho\, d\varphi\, dz$$

$$\boxed{\text{Gradient divergence and curl in cylindrical coordinates}}$$

$$\nabla f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \varphi} \hat{\varphi} + \frac{\partial f}{\partial z} \hat{k}$$

$$\nabla \cdot \bar{v} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho v_\rho) + \frac{1}{\rho} \frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z}$$

$$\nabla \times \bar{v} = \left(\frac{1}{\rho} \frac{\partial v_z}{\partial \varphi} - \frac{\partial v_\varphi}{\partial z} \right) \hat{\rho} + \left(\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho} \right) \hat{\varphi}$$

$$+ \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho v_\varphi) - \frac{\partial v_\rho}{\partial \varphi} \right] \hat{k}$$

$$\Delta f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$