Differential Vector Calculus

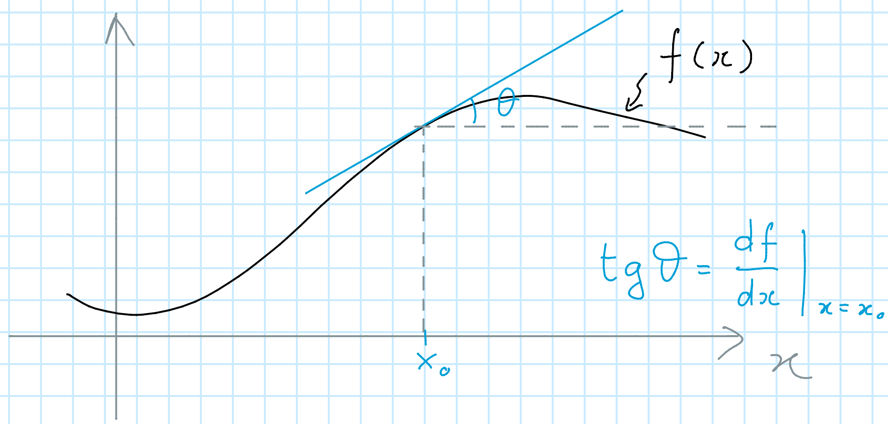
Friday, January 4, 2019 6:12 AM

Ordinary derivatives

A derivative is measure of how rapidly a function f changes if the argument x changes by an infinitesimal amount dx

$$df = \frac{df}{dx}\, dx$$

The derivative at a given point is the slope of f at that point.



Gradient

Definition : $f(x, y, z)$ scalar function

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \underbrace{\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\imath} + \frac{\partial}{\partial z}\hat{k}\right)}_{\equiv\, \nabla} f(x, y, z)$$

$\equiv \nabla$ "del" operator

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The gradient is a vector.

Geometrical interpretation of the gradient

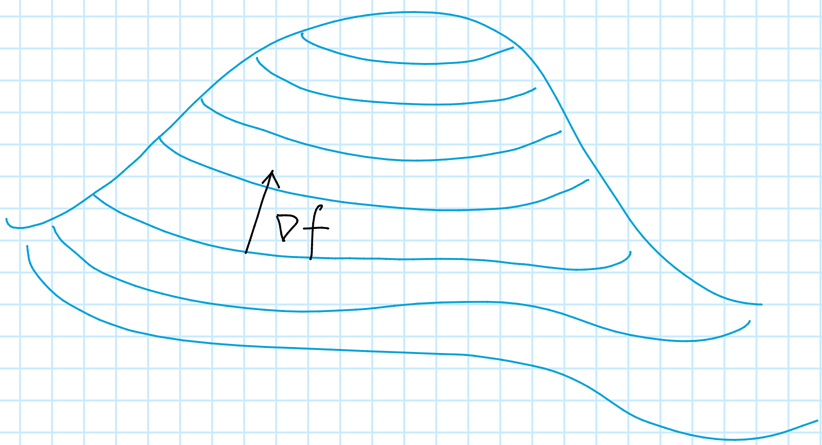
Consider the temperature in a room as a function of the position in the room. The temperature is then a scalar function of three variables T(x,y,z). An infinitesimal temperature change which results from an infinitesimal change in position can be written as

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$$= (\nabla T) \cdot d\bar{\ell} = |\nabla T|\,|d\bar{\ell}| \cos\vartheta$$

If one keeps the magnitude of the gradient fixed, the largest dT is obtained when the angle theta is zero. Therefore the gradient points in the direction of maximum increase of the function T.

A two dimensional example is easier to visualize



$$\nabla f = 0 \longrightarrow \text{stationary point} \begin{cases} \text{max (summit)} \\ \text{min (valley)} \\ \text{saddle point (pass)} \end{cases}$$

Divergence

The divergence is a scalar obtained from the scalar product of a del operator with a vector

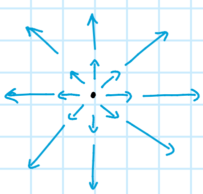
$$\nabla \cdot \bar{v} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left(v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

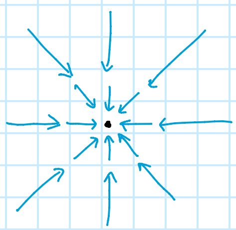
Geometrical interpretation of a divergence

The divergence is a measure of how much a vector spreads out (i.e. diverges) from a given point.

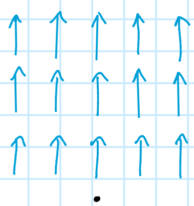
Two dimensional examples



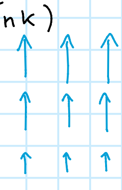
positive div (faucet)



negative div (sink)



zero div



some positive div

Curl

The curl is the cross product of the del operator with a vector function. The curl is a vector

$$\nabla \times \bar{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\imath}\left(\partial_y v_z - \partial_z v_y\right) - \hat{\jmath}\left(\partial_x v_z - \partial_z v_x\right) + \hat{k}\left(\partial_x v_y - \partial_y v_x\right)$$

In Einstein's notation

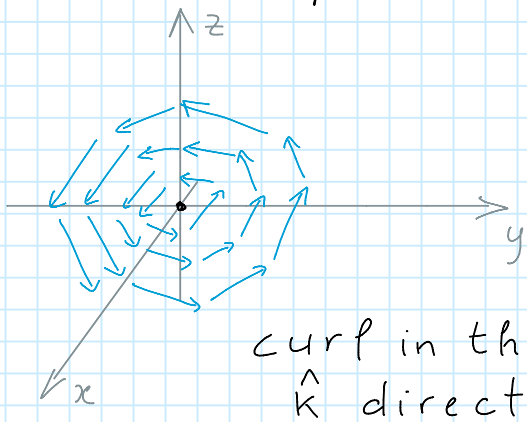
$$\left[\nabla \times \bar{v}\right]_i = \varepsilon_{ijk}\, \partial_j v_k \qquad i,j,k \in \{1,2,3\}$$

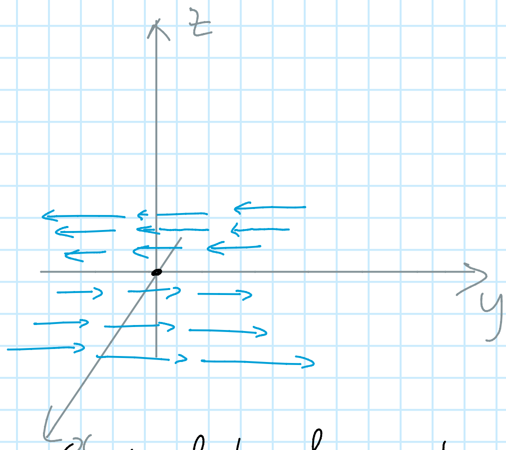
rem there is a summation over repeated indices

Geometrical interpretation of the curl

The curl is a measure of how much the vector v swirls around the point in which the curl is calculated.

Two dimensional examples





curl in the $\hat{k}$ direction (right hand rule)

Product rules

Given two objects, which can be either scalar or a vector, we have two ways to form a scalar and two ways to form a vector

scalars $\quad fg \; , \; \bar{A}\cdot\bar{B}$

vectors $\quad f\bar{A} \; , \; \bar{A}\times\bar{B}$

One can then calculate the gradient of the two scalars

$$\nabla(fg) = f\nabla g + g\nabla f \qquad (i)$$

$$\nabla(\bar{A}\cdot\bar{B}) = \bar{A}\times(\nabla\times\bar{B}) + \bar{B}\times(\nabla\times\bar{A}) + (\bar{A}\cdot\nabla)\bar{B} + (\bar{B}\cdot\nabla)\bar{A} \qquad (ii)$$

One can also calculate the divergence of the two vectors

$$\nabla\cdot(f\bar{A}) = f(\nabla\cdot\bar{A}) + \bar{A}\cdot(\nabla f) \qquad (iii)$$

$$\nabla\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A}) - \bar{A}\cdot(\nabla\times\bar{B}) \qquad (iv)$$

Or the curl of the two vectors

$$\nabla\times(f\bar{A}) = f(\nabla\times\bar{A}) - \bar{A}\times(\nabla f) \qquad (v)$$

$$\nabla\times(\bar{A}\times\bar{B}) = (\bar{B}\cdot\nabla)\bar{A} - (\bar{A}\cdot\nabla)\bar{B} + \bar{A}(\nabla\cdot\bar{B}) - \bar{B}(\nabla\cdot\bar{A}) \qquad (vi)$$

Homework: Check the six relations above by using components and your preferred computer algebra system. Hint: It is often convenient to use components notation. Ex

$$\nabla\cdot(f\bar{A}) = \partial_i(fA_i) = (\partial_i f)A_i + f\,\partial_i A_i = \nabla f\cdot\bar{A} + f\nabla\cdot\bar{A}$$

Second derivatives

By combining gradients, divergence and curl one can build five different kinds of second derivatives

1) Divergence of gradient $\nabla \cdot (\nabla f)$
2) Curl of a gradient $\nabla \times (\nabla f)$
3) Gradient of a divergence $\nabla (\nabla \cdot \bar{v})$
4) Divergence of a curl $\nabla \cdot (\nabla \times \bar{v})$
5) Curl of a curl $\nabla \times (\nabla \times \bar{v})$

Case 1)

$$\nabla \cdot (\nabla f) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$

LAPLACIAN OPERATOR

$$\equiv \nabla^2 f = \Delta f$$

The Laplacian can also be applied to a vector, with the understanding that the operator must be applied to each component of the vector

$$\Delta \bar{v} = (\Delta v_x)\hat{\imath} + (\Delta v_y)\hat{\jmath} + (\Delta v_z)\hat{k}$$

Case 2)

$$\nabla \times (\nabla f) = 0$$

ALWAYS!

To prove this just calculate

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix}$$

Case 3) $\nabla (\nabla \cdot \bar{v})$

Seldom used, it is <u>not</u> the same as the Laplacian

Case 4)

$$\nabla \cdot (\nabla \times \bar{v}) = 0 \qquad \text{ALWAYS!}$$

The relation above is easily proven by writing it out in components.

Case 5)

$$\nabla \times (\nabla \times \bar{v}) = \nabla (\nabla \cdot \bar{v}) - \Delta \bar{v}$$

in fact

$$\nabla \times (\nabla \times \bar{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_y v_z - \partial_z v_y & \partial_z v_x - \partial_x v_z & \partial_x v_y - \partial_y v_x \end{vmatrix}$$

Consider now the x component

$$\partial_y (\partial_x v_y - \partial_y v_x) - \partial_z (\partial_z v_x - \partial_x v_z)$$

$$= \partial_y \partial_x v_y - \partial_y^2 v_x - \partial_z^2 v_x + \partial_z \partial_x v_z \underbrace{- \partial_x^2 v_x + \partial_x^2 v_x}_{\text{add and subtract}}$$

$$= \partial_x (\partial_x v_x + \partial_y v_y + \partial_z v_z) - (\partial_x^2 + \partial_y^2 + \partial_z^2) v_x$$

$$= \partial_x (\nabla \cdot \bar{v}) - \Delta v_x$$

The other components follow a similar pattern