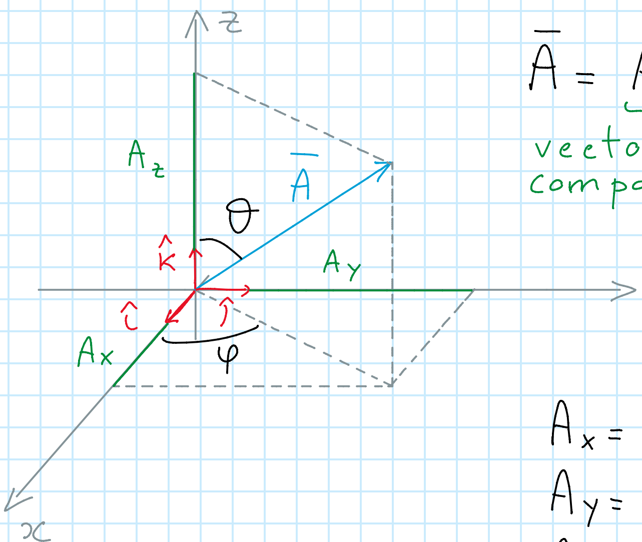
Vectors - Review

Thursday, January 3, 2019 4:46 AM

Students are supposed to know what vectors are, how to sum and subtract them, how to multiply them by a constant, and to know what a scalar and vector product are. Here we review all of these topics in terms of components

Vector components



$$\bar{A} = \underbrace{A_x}_{\text{vector component}} \underbrace{\hat{\imath}}_{\text{unit vector}} + A_y \hat{\jmath} + A_z \hat{k}$$

$$A_x = A \sin\vartheta \cos\varphi$$
$$A_y = A \sin\vartheta \sin\varphi$$
$$A_z = A \cos\vartheta$$

Vector sum

$$\bar{A} \equiv A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}$$
$$\bar{B} \equiv B_x \hat{\imath} + B_y \hat{\jmath} + B_z \hat{k}$$
$$\bar{A} + \bar{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}$$

Vectors times scalar

$$\underset{\text{scalar}}{a} \bar{A} = a A_x \hat{\imath} + a A_y \hat{\jmath} + a A_z \hat{k}$$

Scalar product (dot product)

Vector.vector --> scalar (for example work $W = \bar{F} \cdot \bar{d}$)

$$\bar{A} \cdot \bar{B} = A_x B_x + A_y B_y + A_z B_z$$

since
$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$
$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = \hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = 0$$

therefore

$$A^2 = \bar{A} \cdot \bar{A} = A_x^2 + A_y^2 + A_z^2$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

The equation for the magnitude of the vector A is in agreement with what is implied by the geometric definition of the components given t the beginning of the page.

Vector product (cross product)

Vector X vector --> vector (for example torque $\bar{\tau} = \bar{r} \times \bar{F}$)

$$\bar{A} \times \bar{B} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix} =$$

$$= \hat{i}\,(A_y B_z - A_z B_y) - \hat{j}\,(A_x B_z - A_z B_x) + \hat{k}\,(A_x B_y - A_y B_x)$$

it follows that

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{i} \times \hat{k} = -\hat{j} \qquad \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{j} = -\hat{i}$$

observe that

$$\boxed{\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}}$$

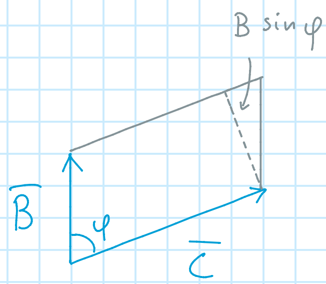
The vector product is not commutative, but anticommutative.

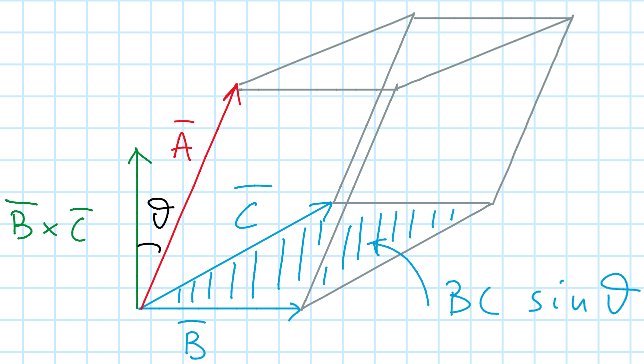
Triple products

One can build three meaningful products between three vectors by using dot and cross products.

Case 1) $\bar{A} \cdot (\bar{B} \times \bar{C})$

$$|\bar{B} \times \bar{C}| = \underbrace{BC \sin\varphi}_{\substack{\text{area of the} \\ \text{parallelogram of} \\ \text{sides } \bar{B} \text{ and } \bar{C}}}$$





$$\bar{A} \cdot (\bar{B} \times \bar{C}) = \underbrace{A \cos\vartheta \, (BC \sin\varphi)}_{\substack{\text{volume of the} \\ \text{parallelepiped} \\ \text{generated by } A, B, C}}$$

Component wise, this can also be written as a determinant

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Since the determinant is invariant with respect to cyclic permutations of the rows one has that

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = \bar{B}\cdot(\bar{C}\times\bar{A}) = \bar{C}\cdot(\bar{A}\times\bar{B})$$

Which also implies that

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = (\bar{A}\times\bar{B})\cdot\bar{C}$$

(swapping of dot and cross)

Since the cross product is anticommutative one also finds that

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = -\bar{A}\cdot(\bar{C}\times\bar{B}) = -\bar{C}\cdot(\bar{B}\times\bar{A}) = -\bar{B}\cdot(\bar{A}\times\bar{C})$$

Case 2) $\bar{A}\times(\bar{B}\times\bar{C})$

This product can be simplified by the rule

$$\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}\,(\bar{A}\cdot\bar{C}) - \bar{C}\,(\bar{A}\cdot\bar{B})$$

One can check that this rule applies by looking at the equation above in components

$$(\bar{B}\times\bar{C})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} B_j C_k = \varepsilon_{ijk} B_j C_k$$

antisymmetric tensor

repeated index (Einstein) notation

$$\left(\bar{A}\times(\bar{B}\times\bar{C})\right)_i = \varepsilon_{ilm} A_l (\bar{B}\times\bar{C})_m = \varepsilon_{ilm} A_l \varepsilon_{mjk} B_j C_k$$

$$= \underbrace{\varepsilon_{mil}\,\varepsilon_{mjk}}_{\delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj}} A_l B_j C_k$$

$$= A_l B_i C_l - A_l C_i B_l$$

$$= B_i (\bar{A}\cdot\bar{C}) - C_i (\bar{A}\cdot\bar{B})$$

Observe that

$$(\bar{A}\times\bar{B})\times\bar{C} = -\bar{C}\times(\bar{A}\times\bar{B})$$

$$= -\bar{A}(\bar{B}\cdot\bar{C}) + \bar{B}(\bar{A}\cdot\bar{C})$$

So (A X B) X C is completely different vector with respect to A X (B X C) , the cross product is not associative.