Maxwell's equations in integral form

Saturday, September 1, 2018 8:45 AM

ME are often expressed in integral form

Gauss' law for E

Integrate over a finite volume V bound by a closed surface ∂V

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \longrightarrow \int_V \nabla \cdot \bar{E}\, d^3r \overset{\text{GAUSS}}{=} \underbrace{\int_{\partial V} \bar{E} \cdot d\bar{s}}_{\text{FLUX } \phi_E(\partial V, t)} = \frac{1}{\varepsilon_0} \underbrace{\int_V \rho\, d^3r}_{\text{TOTAL CHARGE } Q}$$

$$\phi_E(\partial V, t) = \frac{Q}{\varepsilon_0}$$

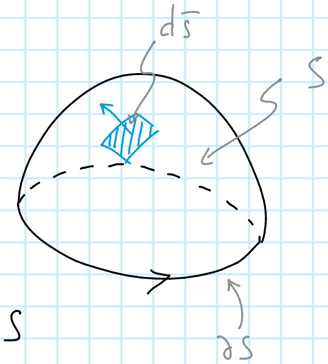
The flux of E through the surface of V is proportional to Q

Gauss' law for B

$$\nabla \cdot \bar{B} = 0 \longrightarrow \int_V \nabla \cdot \bar{B}\, d^3r = \underbrace{\int_{\partial V} \bar{B} \cdot d\bar{s}}_{\phi_B(\partial V, t)} = 0 \longrightarrow \phi_B = 0$$

The flux of B over a closed surface is zero, magnetic field lines are loops (closed lines)

Ampere-Maxwell's law



$$\nabla \times \bar{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\bar{E} + \mu_0 \bar{J}$$

Integrate over a surface S bound by a closed line ∂S

$$\int_S (\nabla \times \bar{B}) \cdot d\bar{s} \overset{\text{STOKES}}{=} \oint_{\partial S} \bar{B} \cdot d\bar{\ell} = \mu_0 \underbrace{\int_S d\bar{s} \cdot \left(\frac{\partial \bar{E}}{\partial t}\varepsilon_0 + \bar{J}\right)}_{I(S,t)}$$

CURRENT THROUGH THE SURFACE S

$$\oint_{\partial S} \bar{B} \cdot d\bar{l} = \mu_0 I$$

I is the total current through the surface S. Notice that part of I comes from the Maxwell's displacement current $\varepsilon_0 \frac{\partial \bar{E}}{\partial t}$

Faraday's induction law

$$\nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t}$$

As for the Ampere Maxwell's case, integrate over a surface S bound by a closed line ∂S

$$\int_S (\nabla \times \bar{E}) \cdot d\bar{s} = \oint_{\partial S} \bar{E} \cdot d\bar{l} = -\int_S \left(\frac{\partial}{\partial t} \bar{B} \right) \cdot d\bar{s} \qquad (I)$$

If the surface S does not change in time, one can exchange integration and time derivative to find

$$\underbrace{\oint_{\partial S} \bar{E} \cdot d\bar{l}}_{\mathcal{E} \text{ ELECTROMOTIVE FORCE (emf)}} = - \frac{d}{dt} \underbrace{\int_S \bar{B} \cdot d\bar{s}}_{\phi_B (S,t)} \longrightarrow \mathcal{E} = -\frac{d}{dt} \phi_B$$

The result remains valid also if S changes in time, but one needs to be careful with vector calculus and with the definition of e.m.f.

Let's start by considering the following theorem valid for a vector B (proof in Zangwill 1.4.5)

$$\frac{d}{dt} \int_{S(t)} \bar{B} \cdot d\bar{s} = \int_{S(t)} d\bar{s} \cdot \left\{ \frac{\partial \bar{B}}{\partial t} + \bar{v}_c \, (\nabla \cdot \bar{B}) - \nabla \times (\bar{v}_c \times \bar{B}) \right\}$$

In the equation above $\bar{v}_c$ is the velocity of a given differential element $d\bar{S}$ of the surface.

If we identify the vector B with the magnetic field, the relation above simplifies because the divergence of B is zero.

$$\frac{d}{dt} \int_{S(t)} \bar{B} \cdot d\bar{s} = \int_{S(t)} d\bar{s} \cdot \frac{\partial \bar{B}}{\partial t} - \int_{S(t)} \nabla \times (\bar{v}_c \times \bar{B}) \cdot d\bar{s}$$

$$\frac{d}{dt}\int_{S(t)} \bar{B}\cdot d\bar{s} = \int_{S(t)} d\bar{s}\cdot\frac{\partial \bar{B}}{\partial t} - \oint_{\partial S(t)} (\bar{v}_c \times \bar{B})\cdot d\bar{\ell}$$

$$\int_{S(t)} d\bar{s}\cdot\frac{\partial \bar{B}}{\partial t} = \frac{d}{dt}\int_{S(t)} \bar{B}\cdot d\bar{s} + \oint_{\partial S(t)} (\bar{v}_c \times \bar{B})\cdot d\bar{\ell} \qquad (\mathrm{II})$$

Now replace (II) in (I)

$$\oint_{\partial S(t)} \bar{E}\cdot d\bar{\ell} = -\frac{d}{dt}\int_{S(t)} \bar{B}\cdot d\bar{s} - \oint_{\partial S(t)} (\bar{v}_c \times \bar{B})\cdot d\bar{\ell}$$

$$\oint_{\partial S(t)} (\bar{E} + \bar{v}_c \times \bar{B})\cdot d\bar{\ell} = -\frac{d}{dt}\underbrace{\int_{S(t)} \bar{B}\cdot d\bar{s}}_{\phi_B} \qquad (\mathrm{III})$$

If ∂S is assumed to coincide with a loop of conducting wire which can change shape in time, $\bar{v}_c$ is the local velocity of the heavy ions which give the material mass and rigidity. Electrons carrying current in the wire move with respect to the ions with a drift velocity

$$\underbrace{\bar{v}_e(\bar{r},t)}_{e^- \text{ velocity}} \equiv \bar{v}_c(\bar{r},t) + \underbrace{\bar{v}_d(\bar{r},t)}_{\text{drift velocity}}$$

However the drift velocity is parallel to the line element dl, therefore

$$(\bar{v}_d \times \bar{B})\cdot d\bar{\ell} = 0$$

One can then replace the drift velocity with the electron velocity in (III)

$$\underbrace{\oint_{\partial S(t)} d\bar{\ell}\cdot(\bar{E} + \bar{v}_e \times \bar{B})}_{\text{FARADAY'S ELECTROMOTIVE FORCE}} = -\frac{d}{dt}\phi_B$$