Solution of the wave equation

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Our goal is now to solve the wave equation satisfied by E and B. In addition one needs to make sure that the solutions still satisfy Maxwell's equations. In this way one will find a relationship between the fields E and B.

Plane waves

We start by looking at a special class of solutions, in which the wave propagates along one direction, say the x axis, and the fields do not depend on the other two space coordinates. The solutions in this class are referred to as plane waves.

Let's analyze the divergence of E in this context

$$\nabla \cdot \bar{E} = 0 \longrightarrow \partial_x E_x(x,t) + \underbrace{\cancel{\partial_y E_y(x,t)}}_{0} + \underbrace{\cancel{\partial_z E_z(x,t)}}_{0} = 0$$

$$\frac{\partial}{\partial x} E_x(x,t) = 0 \qquad E_x = \text{const. w.r.t } x = E_x(t)$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\text{similarly} \qquad \nabla \cdot \bar{B} = 0 \longrightarrow B_x \equiv B_x(t)$$

$$\nabla \times \bar{B} = \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t} \longrightarrow (\nabla \times \bar{B})_x = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}$$

$$\partial_y B_z(x,t) - \partial_z B_y(x,t) = 0 = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} \qquad \boxed{E_x = \text{const.}}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}_x}{\partial t} \longrightarrow (\nabla \times \bar{E})_x = -\frac{\partial B_x}{\partial t}$$

$$\partial_y E_z(x,t) - \partial_z E_y(x,t) = 0 = -\frac{\partial B_x}{\partial t} \qquad \boxed{B_x = \text{const.}}$$

One can always add a posteriori constant contribution to the solutions of Maxwell's equation. Therefore we are allowed to choose E_x and B_x simply equal to zero.
The solution that we are looking for is therefore of the form

$$\bar{E}(x,t) = E(x,t)\,\hat{j}$$

That means that we are looking for a solution where the electric field remains along the same line all the time (i.e. We are looking for a linearly polarized wave). The wave equation then becomes

$$\boxed{\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \Delta E = 0}$$

The most general solution of the equation above has the form

$$E(x,t) = \underbrace{f(x-ct)}_{\substack{\text{wave advancing} \\ \text{toward } x\to+\infty \\ \text{with speed } c}} + \underbrace{g(x+ct)}_{\substack{\text{wave advancing} \\ \text{toward } x\to-\infty}}$$

The generic solution outlined above can be written as a superposition of sinusoidal waves of fixed frequency. These waves are called monochromatic waves. A general monochromatic wave can be written as

$$E = E_0 \sin\left[\omega\left(\frac{x}{c} - t\right) + \phi\right]$$

set $\phi = 0$ for the time being

$$\boxed{E = E_0 \sin\left[kx - \omega t\right]}$$

$k \equiv \frac{\omega}{c}$ = wave number

rem

$\omega \equiv 2\pi\nu$ $\qquad$ ω = angular frequency

ν (or f) = frequency

$\lambda \equiv \frac{2\pi}{k}$ $\qquad$ λ = wavelength

visible light $3.9\times10^{-7}\,m < \lambda < 7\times10^{-7}\,m$

E_0 = amplitude

In order to determine B once E is known we turn again to the equation for the curl of E

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \longrightarrow \hat{k}\, \partial_x E_y = -\frac{\partial B_z}{\partial t}\hat{k}$$

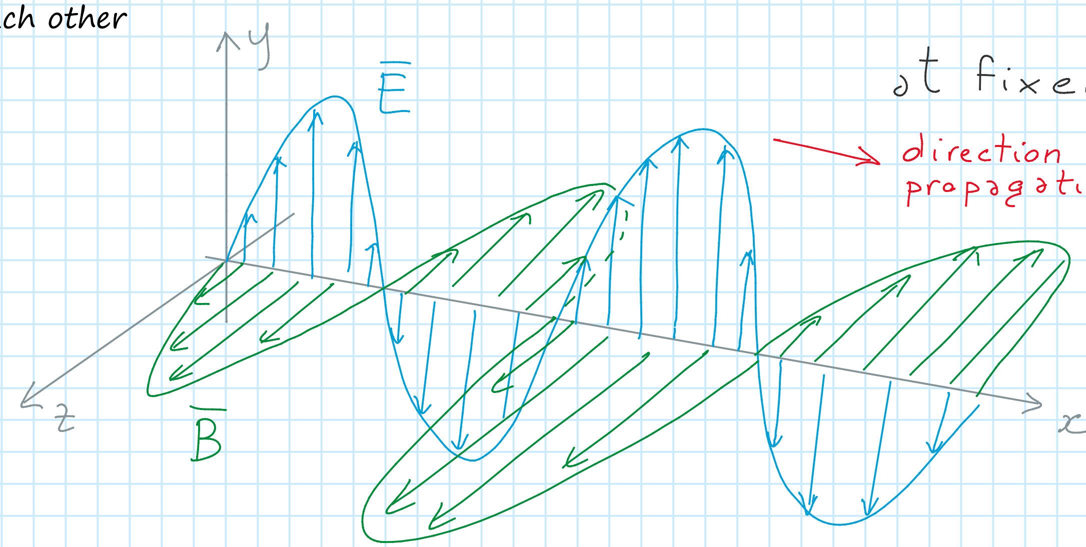
$$\bar{B} = B(x,t)\,\hat{k}$$

B only has a z component

$$\frac{\partial B}{\partial t} = -\frac{\partial}{\partial x}E = -k E_0 \cos(kx - \omega t)$$

$$\boxed{B(x,t) = \frac{E_0}{c}\sin(kx - \omega t)}$$

The electric and magnetic fields oscillate in phase and they are perpendicular to each other



Since Maxwell's equations are linear in E and B, any linear combination of sinusoidal functions is still a valid solution of Maxwell equations. By using Fourier analysis one can build waves of essentially arbitrary shape. But be careful, the shape of the wave might be not constant in time.

A useful notation can be introduced at this point, we can define complex electric and magnetic fields with the understanding that the physical fields are either the real of the imaginary part of the fields

$$\bar{E} = E_0\,\hat{j}\,e^{i(kx - \omega t)} \qquad \bar{B} = \frac{E_0}{c}\,\hat{k}\,e^{i(kx-\omega t)}$$

$$\bar{E} = E_0 \hat{j} e^{i(kx-\omega t)} \qquad \bar{B} = \frac{E_0}{c} \hat{k} e^{i(kx-\omega t)}$$

It is convenient to do this because it is easier to manipulate exponentials than to manipulate sine and cosine functions.

It is important to point out that if E_1 and E_2 are two complex solutions of Mazwell's equations, any linear combination of these two fields is still a solution of Maxwell's equations. This is a consequence of the linearity of the equations.