Force and energy of a magnetic dipole in a magnetic field

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We want to find out what is the force acting on a current density j placed in a magnetic field B. The force acting on the current density can be obtained starting from Lorentz force

$$\bar{F} = \int_V d^3r \; \bar{j}(\bar{r}) \times \bar{B}(\bar{r})$$

We now assume that the current is localized in a small region surrounding the point r = R and that the magnetic field B varies slowly around that point. One can then Taylor expand the magnetic field as follows

$$\bar{B}(\bar{r}) = \bar{B}(\bar{R}) + \left[(\bar{r}-\bar{R})\cdot\nabla\right]\bar{B}(\bar{R}) + \dots$$

By inserting this expansion in the equation for the Lorentz force one finds

$$\bar{F} = \left[\int_V d^3r \; \bar{j}(\bar{r})\right] \times \bar{B}(\bar{R})$$

$$+ \int_V d^3r \; \bar{j}(\bar{r}) \times \left[(\bar{r}\cdot\nabla)\bar{B}(\bar{R})\right]$$

$$- \left[\int_V d^3r \; \bar{j}(\bar{r})\right] \times \left[(\bar{R}\cdot\nabla)\bar{B}(\bar{R})\right]$$

However, the first and last integral vanish because

$$\int_V d^3r \; \bar{j}(\bar{r}) = 0$$

(Remember

$$\partial_\ell (j_\ell r_i) = \underbrace{(\partial_\ell j_\ell)}_{\nabla \cdot \bar{j} = 0} r_i + j_i = j_i$$

$$\hookrightarrow \int_V d^3r\, \bar{j}(\bar{r}) = \int_V d^3r\, \partial_\ell (j_\ell r_i)\, \hat{\imath}_i = 0$$

The last step is due to the fact that the integral is over a total derivative and therefore can be rewritten as a surface term. The surface term vanishes if we are dealing with a localized current density and we integrate over a volume that includes it completely.) Therefore

$$\bar{F} = \int_V d^3r\, \bar{j}(\bar{r}) \times \left[(\bar{r} \cdot \underbrace{\nabla') \bar{B}(\bar{r}')}_{\nabla' \equiv \frac{\partial}{\partial x_i'} \hat{\imath}_i} \right] \Big|_{\bar{r}' = \bar{R}}$$

At this stage it is necessary to show that the integrand can be rewritten as

$$\bar{j}(\bar{r}) \times \left[(\bar{r} \cdot \nabla') \bar{B}(\bar{r}') \right] = -\nabla' \times \left[(\bar{r} \cdot \bar{B}(\bar{r}')) \bar{j}(\bar{r}) \right]$$

It is convenient to start the proof by writing the integrand in components

$$\left\{ \bar{j} \times \left[(\bar{r} \cdot \nabla') \bar{B} \right] \right\}_i = \varepsilon_{i\ell m}\, j_\ell \left[(r_p\, \partial_p') B_m \right] \qquad (\mathrm{I})$$

We want to show that the above is equal to

$$\left\{ -\nabla' \times \left[(\bar{r} \cdot \bar{B}) \bar{j} \right] \right\}_i = -\varepsilon_{im\ell}\, \partial_m' \left[r_p B_p j_\ell \right]$$

$$= \varepsilon_{i\ell m}\, j_\ell\, r_p\, \partial_m' B_p \qquad (\mathrm{II})$$

$$(\mathrm{I}) - (\mathrm{II}) = \varepsilon_{ilm}\, j_l\, r_p \Big[\underbrace{\partial'_p B_m - \partial'_m B_p}_{\text{components of } \nabla \times \bar{B} = 0} \Big] = 0$$

One can then conclude that

since we are looking at the force applied by a given external $\bar{B}$ on $\bar{j}$

$$\bar{F} = -\nabla' \times \int_V d^3r \, \left(\bar{r} \cdot \bar{B}(\bar{r}') \right) \bar{j}(\bar{r}) \Big|_{\bar{r}' = \bar{R}}$$

By using the identity in eq (3.22) of the book by Professor D. Tong

$$\int d^3r' \, \bar{j} \, (\bar{r} \cdot \bar{r}') = \frac{1}{2} \bar{r} \times \int d^3r' \, \bar{j} \times \bar{r}'$$

$\bar{r}' \longrightarrow \bar{r}$ $\quad$ $\bar{r} \longrightarrow \bar{B}$

$$\int d^3r \, \bar{j} \, (\bar{B} \cdot \bar{r}) = \frac{1}{2} \bar{B} \times \int d^3r \, \bar{j} \times \bar{r}$$

$$= -\bar{B} \times \left(\frac{1}{2} \int d^3r \, \bar{r} \times \bar{j} \right)$$

$$= -\bar{B} \times \bar{m}$$

MAGNETIC DIPOLE MOMENT

$$\bar{F} = +\nabla \times \left(\bar{B}(\bar{r}) \times \bar{m} \right)$$

(at this stage only B depends on the position so one can drop the prime with the understanding that the derivatives are evaluated at the location of the dipole moment)

$$\bar{F} = \nabla \times (\bar{B} \times \bar{m}) = (\bar{m} \cdot \nabla)\bar{B} - \cancelto{0}{(\bar{B} \cdot \nabla)\bar{m}} + \cancelto{0}{\bar{B}(\nabla \cdot \bar{m})}$$

$$- \bar{m}\,\underbrace{(\nabla \cdot \bar{B})}_{=0}$$

($\bar{m}$ does not depend on $\bar{r}$)

$$= \nabla(\bar{B} \cdot \bar{m}) - \bar{m} \times \underbrace{(\nabla \times \bar{B})}_{=0}$$

$$- \cancelto{0}{\bar{B} \times (\nabla \times \bar{m})} - \cancelto{0}{(\bar{B} \cdot \nabla)\bar{m}}$$

$$= \nabla(\bar{B} \cdot \bar{m})$$

The force applied to a magnetic dipole by a magnetic field is therefore the gradient of the function

$$U = -\bar{B} \cdot \bar{m}$$

The function U is therefore the energy of the dipole in the magnetic field. The energy is minimal if the dipole is aligned to the magnetic field. The magnetic field exerts a torque on the dipole trying to align it to the field.

Observe that the force on the dipole is minus the gradient of the energy. This means that is B is constant, there is no force acting on the dipole, because the magnetic dipole is a constant vector and if B is constant its derivatives with respect to the space coordinates are zero.

Force between two dipoles

Here we combine the equation for the magnetic field surrounding a dipole with the equation for the force applied by a magnetic field on a dipole discussed above:

$$\bar{B}(\bar{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\bar{m}_1 \cdot \hat{r})\hat{r} - \bar{m}_1}{r^3}\right)$$

$$\overline{F} = \nabla\left(\overline{B}\cdot\overline{m}_2\right)$$

$$= \frac{\mu_0}{4\pi}\nabla\left(\frac{3\,(\overline{m}_1\cdot\hat{r})(\overline{m}_2\cdot\hat{r}) - \overline{m}_1\cdot\overline{m}_2}{r^3}\right)$$

$$\nabla\frac{1}{r^3} = -\frac{3}{r^4}\hat{r}$$

$$\nabla\frac{(\overline{m}_1\cdot\hat{r})(\overline{m}_2\cdot\hat{r})}{r^3} = \nabla\frac{(\overline{m}_1\cdot\overline{r})(\overline{m}_2\cdot\overline{r})}{r^5}$$

$$= \partial_i\frac{m_{1\ell}\,x_\ell\,m_{2p}\,x_p}{r^5}\,\hat{i}_i = \frac{1}{r^5}\left(m_{1i}\,\overline{m}_2\cdot\overline{r} + m_{2i}\,\overline{m}_1\cdot\overline{r}\right)\hat{i}_i$$

$$- 5\frac{\hat{r}}{r^6}(\overline{m}_1\cdot\overline{r})(\overline{m}_2\cdot\overline{r})$$

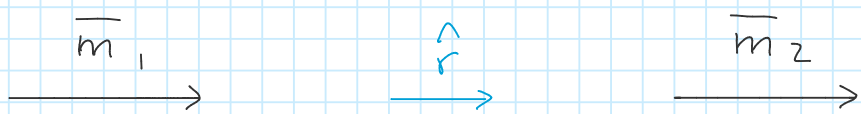
$$= \frac{\overline{m}_1(\overline{m}_2\cdot\hat{r})}{r^4} + \frac{\overline{m}_2(\overline{m}_1\cdot\hat{r})}{r^4} - 5\frac{(\overline{m}_1\cdot\hat{r})(\overline{m}_2\cdot\hat{r})}{r^4}\hat{r}$$

$$\overline{F} = \frac{3\mu_0}{4\pi}\frac{1}{r^4}\Big[\overline{m}_1(\overline{m}_2\cdot\hat{r}) + \overline{m}_2(\overline{m}_1\cdot\hat{r})$$

$$- 5(\overline{m}_1\cdot\hat{r})(\overline{m}_2\cdot\hat{r})\hat{r} + \overline{m}_1\cdot\overline{m}_2\,\hat{r}\Big]$$

Notice that the force is NOT directed along r.

Special cases

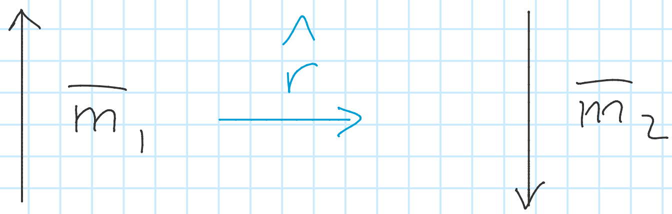


$$\overline{F} = \frac{3\mu_0}{4\pi}\frac{\hat{r}}{r^4}\left[-2m_1\,m_2\right]$$

ATTRACTIVE FORCE

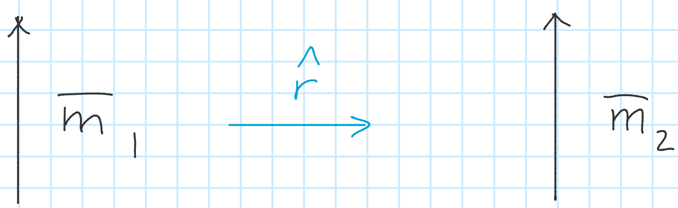
$$\bar{F} = \frac{3\mu_0}{4\pi}\,\frac{\hat{r}}{r^4}\left[2m_1\,m_2\right]$$

REPULSIVE FORCE



$$\bar{F} = \frac{3\mu_0}{4\pi}\,\frac{\hat{r}}{r^4}\left[-\,m_1 m_2\right]$$

ATTRACTIVE FORCE



$$\bar{F} = \frac{3\mu_0}{4\pi}\,\frac{\hat{r}}{r^4}\left[m_1 m_2\right]$$

REPULSIVE FORCE