Auxiliary field H

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We established that a magnetization M creates bound currents within the material and on the surface

$$\bar{J}_b = \nabla \times \bar{M} \qquad \bar{K}_b = \bar{M} \times \hat{n}$$

Let's now look at the combined effect of the free current (due typically to a battery) and the bound current (due to magnetization)

$$\bar{J} = \bar{J}_b + \bar{J}_f$$

One can then use the relation above in Ampere's law

$$\nabla \times \bar{B} = \mu_0 (\bar{J}_b + \bar{J}_f) \longrightarrow \frac{1}{\mu_0}(\nabla \times \bar{B}) = \bar{J}_f + \bar{J}_b = \bar{J}_f + \nabla \times \bar{M}$$

$$\nabla \times \left(\frac{1}{\mu_0} \bar{B} - \bar{M} \right) = \bar{J}_f$$

One can then define the auxiliary field H

$$\boxed{\bar{H} \equiv \frac{1}{\mu_0} \bar{B} - \bar{M}}$$

So that

$$\boxed{\nabla \times \bar{H} = \bar{J}_f} \qquad \boxed{\oint \bar{H} \cdot d\bar{\ell} = I_{free}}$$

These equations are useful because they involve only free currents. However, one should observe that there is no complete analogy between B and H, since

$$\nabla \cdot \bar{B} = 0 \qquad \text{but} \qquad \nabla \cdot \bar{H} = -\nabla \cdot \bar{M} \neq 0$$

(in general)

In particular, if there is no free current

$$\nabla \times \bar{H} = 0$$

However, one cannot assume that H = 0. Consider for example a bar magnet, where there is no free current and therefore the curl of H is zero everywhere. If one assumes (making a mistake) that H = 0, one would be forced to conclude that B is zero outside the magnet, while inside the magnet it is

$$\bar{B} = \mu_0 \bar{M}$$

This conclusion is obviously absurd.

Boundary conditions for H

One can write boundary conditions for the field H near a surface with surface current density K.

$$B^{\perp}_{above} - B^{\perp}_{below} = 0$$

$$\frac{1}{\mu_0} B^{\perp}_{above} - \frac{1}{\mu_0} B^{\perp}_{below} - M^{\perp}_{above} + M^{\perp}_{below} = - M^{\perp}_{above} + M^{\perp}_{below}$$

$$\boxed{H^{\perp}_{above} - H^{\perp}_{below} = -\left(M^{\perp}_{above} - M^{\perp}_{below}\right)}$$

$$\bar{B}^{\parallel}_{above} - \bar{B}^{\parallel}_{below} = \mu_0 \left(\bar{K} \times \hat{n}\right)$$

$$\frac{1}{\mu_0} \bar{B}^{\parallel}_{above} - \bar{M}^{\parallel}_{above} - \left(\frac{1}{\mu_0} \bar{B}^{\parallel}_{below} - \bar{M}^{\parallel}_{below}\right)$$

$$= \bar{K} \times \hat{n} - \left(\bar{M}^{\parallel}_{above} - \bar{M}^{\parallel}_{below}\right)$$

In order to proceed further, one needs to derive the identity

$$\bar{M} - \hat{n}\, \bar{M} \cdot \hat{n} = - \bar{K}_b \times \hat{n} \qquad (I)$$

(rem: $\hat{n}$ unit vector $\perp$ to the surface where there is a magnetization $\bar{M}$)

In order to prove (i) let's consider the i-th component of the cross product

$$(\bar{K}_b \times \hat{n})_i = \left[(\bar{M} \times \hat{n}) \times \hat{n} \right]_i = \varepsilon_{ijk}\, \varepsilon_{jpq} M_p n_q n_k$$

$\bar{K}_b \equiv \bar{M} \times \hat{n}$

$$= -\varepsilon_{jik}\, \varepsilon_{jpq} M_p n_q n_k$$

$$= -\left(\delta_{ip}\delta_{kq} - \delta_{iq}\delta_{kp} \right) M_p n_q n_k$$

$$= -M_i n_k n_k + n_i M_k n_k$$

$$= -M_i \hat{n}^2 + n_i \bar{M}\cdot\hat{n} = \left[-\bar{M} + \hat{n}\, \bar{M}\cdot\hat{n} \right]_i$$

$$\bar{M} - \underbrace{\hat{n}\, \bar{M}\cdot\hat{n}}_{\equiv \bar{M}_\perp} = -\bar{K}_b \times \hat{n}$$

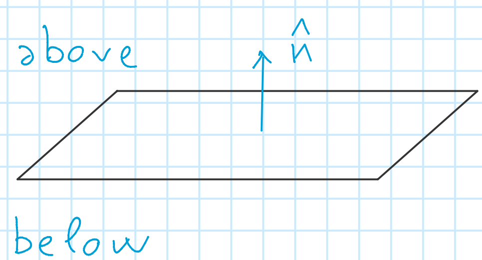
comp. of $\bar{M}$ parallel to $\hat{n}$

$$\bar{M} - \bar{M}_\perp \equiv \bar{M}^{//} = -\bar{K}_b \times \hat{n}$$

this is $\perp$ to $\hat{n}$ and consequently parallel to the interface

The total k_b at the interface of two materials is the sum of the surface currents due to the magnetization in the material above and the material below the interface.

$$\bar{K}_b = \bar{K}_{above} + \bar{K}_{below}$$



$$\overline{M}^{\parallel}_{below} = - \overline{K}_{below} \times \hat{n}$$

$$\overline{M}^{\parallel}_{above} = - \overline{K}_{above} \times \underbrace{(-\hat{n})}_{\text{unit vector pointing out of the "above" material}} = + \overline{K}_{above} \times \hat{n}$$

One can finally connect with the equation that was written above

$$\underbrace{\frac{1}{\mu_0} \overline{B}^{\parallel}_{above} - \overline{M}^{\parallel}_{above}}_{\overline{H}^{\parallel}_{above}} - \Big(\underbrace{\frac{1}{\mu_0} \overline{B}^{\parallel}_{below} - \overline{M}^{\parallel}_{below}}_{\overline{H}^{\parallel}_{below}} \Big)$$

$$= - \left(\overline{M}^{\parallel}_{above} - \overline{M}^{\parallel}_{below} \right) + \overline{K} \times \hat{n}$$

$$\overline{H}^{\parallel}_{above} - \overline{H}^{\parallel}_{below} = - \left(\overline{K}_{above} \times \hat{n} + \overline{K}_{below} \times \hat{n} \right)$$
$$+ \overline{K} \times \hat{n}$$

$$\overline{H}^{\parallel}_{above} - \overline{H}^{\parallel}_{below} = \left(\overline{K} - \overline{K}_b \right) \times \hat{n} = \overline{K}_f \times \hat{n}$$

$$\boxed{\overline{H}^{\parallel}_{above} - \overline{H}^{\parallel}_{below} = \overline{K}_f \times \hat{n}}$$

Observe: One can obtain the equation above directly by applying the theorem for the circulation of H

$$\oint_C \overline{H} \cdot d\overline{l} = I_{free}$$