Separation of variables - spherical coordinates - azimuthal symmetry

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When boundary conditions are fixed on a spherical surface, it is more convenient to start from the Laplace equation written in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right)+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial \varphi}{\partial \vartheta}\right)+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 \varphi}{\partial \phi^2}=0$$

We look here only at the special case of problems with azimuthal symmetry, in which the potential does not depend on the azimuthal angle phi. (One can also deal with the more complicated case in which there isn't an azimuthal symmetry. That case involves spherical harmonic functions and is discussed in a graduate course.)

With azimuthal symmetry Laplace's equation simplifies to

$$\boxed{\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right)+\frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta\frac{\partial \varphi}{\partial \vartheta}\right)=0}$$

LAPLACE'S EQUATION WITH AZIMUTHAL SYMMETRY

We look for a factored solution of the form

$$\varphi(r,\vartheta)=R(r)\,\Theta(\vartheta)$$

By plugging this Ansatz in the Laplace's equation and then dividing everything by the potential, one finds

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{d\Theta}{d\vartheta}\right)=0$$

The equation above is in reality the sum of two ordinary differential equations

$$\boxed{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\ell(\ell+1)}$$

$\ell(\ell+1)$
$= \text{const}$

$$\frac{1}{\Theta \sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{d\Theta}{d\vartheta}\right) = -\ell(\ell+1)$$

The reason for choosing the separation constant equal to l(l+1) becomes obvious when one solves the radial equation

$$\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right) = \ell(\ell+1)\,R$$

The solution for this equation is

$$\boxed{R(r) = A\,r^{\ell} + \frac{B}{r^{\ell+1}}}$$

One can easily check that the function above does satisfy the equation

$$\frac{dR}{dr} = \ell A\,r^{\ell-1} - (\ell+1)\,\frac{B}{r^{\ell+2}}$$

$$r^2\frac{dR}{dr} = \ell A\,r^{\ell+1} - (\ell+1)\,\frac{B}{r^{\ell}}$$

$$\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right) = \ell(\ell+1)\,A\,r^{\ell} + \ell(\ell+1)\,\frac{B}{r^{\ell+1}}$$

$$= \ell(\ell+1)\,R \qquad \checkmark$$

The angular equation is more complicated and its solutions (for integer l) is given by the Legendre polynomials which we already encountered.

The most general solution of Laplace equation in the azimuthal symmetry case is then

$$\varphi(r,\vartheta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^{\ell} + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\vartheta)$$

There are a few cases which are worth discussing in more detail

Inside a hollow sphere

Let's consider the class of problems in which the potential is specified on the surface of a hollow sphere of radius R and one is asked to find the potential inside the sphere.

In addition, we stick to the case of azimuthal symmetry. One should then set B_l = 0 in the previous general solution since the potential cannot blow up in the center of the sphere, which is empty.

$$\varphi(r,\vartheta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\vartheta)$$

With the additional boundary condition

$$\varphi(R,\vartheta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\vartheta) \equiv \underbrace{V(\vartheta)}_{\text{Known potential on the sphere}}$$

The coefficients in the expansion can be fixed by using the orthogonality of the Legendre polynomials

$$\int_{-1}^{1} P_m(x)\,\varphi(R,x)\,dx = \sum_{\ell=1}^{\infty} A_\ell R^\ell \underbrace{\int_{-1}^{1} P_m(x)\,P_\ell(x)\,dx}_{\frac{2}{2m+1}\delta_{\ell m}}$$

$$A_m = \frac{2m+1}{2R^m}\int_{-1}^{1} dx\; P_m(x)\,\varphi(R,x)$$

$$= \frac{2m+1}{2R^m}\int_{0}^{\pi} d\vartheta\,\sin\vartheta\,P_m(\cos\vartheta)\,V(\vartheta)$$

The integrals to fix the coefficients A are not always easy to calculate. Sometimes however one can even fix the coefficients A by eye. Consider the case in which

$$V(\vartheta) = K \sin^2\frac{\vartheta}{2} = \frac{K}{2}(1-\cos\vartheta) = \frac{K}{2}\left[P_0(\cos\vartheta) - P_1(\cos\vartheta)\right]$$

One sees immediately that

$$A_0 = \frac{K}{2}, \quad A_1 = -\frac{K}{2R}, \quad A_\ell = 0 \quad \text{for } \ell = 2,3,4,\ldots$$

Consequently

$$\varphi(r,\vartheta) = \frac{K}{2}\left[P_0 - \frac{r}{R}P_1(\cos\vartheta)\right] = \frac{K}{2}\left(1 - \frac{r}{R}\cos\vartheta\right)$$

Outside a sphere

We consider now the case in which the potential is again specified on the surface of a sphere of radius R, but one wants to find the potential in the space outside the sphere. Again, we consider a case that shows azimuthal symmetry. The potential should die out at infinity, so that the general solution for this situation can be written as

$$\varphi(r,\vartheta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\vartheta)$$

In addition we impose the boundary condition

$$\varphi(R,\vartheta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\vartheta) = \underbrace{V(\vartheta)}_{\text{potential on the sphere (known)}}$$

One can use once more the orthogonality of Legendre's polynomials to find

$$\int_0^\pi d\vartheta\, P_m(\cos\vartheta)\, V(\vartheta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} \underbrace{\int_0^\pi P_\ell(\cos\vartheta)\, P_m(\cos\vartheta)\, d\cos\vartheta}_{\frac{2}{2m+1}\delta_{\ell m}}$$

$$B_m = \frac{2m+1}{2} R^{m+1} \int_0^{\pi} d\cos\theta \; P_m(\cos\theta) \; V(\theta)$$