Fourier Series

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A complete set of functions in the interval [0,a] is given by

$$g_n(x) = \sin\left(\frac{n\pi}{a}x\right)$$

The functions are orthogonal

$$\int_0^a dx \ \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right) = 0 \qquad \text{if } n \neq m$$

Proof

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\int_0^a dx \ \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right) =$$
$$= \frac{1}{2}\left\{\int_0^a dx \ \cos\left[(n-m)\frac{\pi}{a}x\right] - \int_0^a dx \ \cos\left[(n+m)\frac{\pi}{a}x\right]\right\}$$
$$= \frac{1}{2}\frac{a}{(n-m)\pi}\int_0^{(n-m)\pi} du \ \cos u - \frac{1}{2}\frac{a}{(n+m)\pi}\int_0^{(n+m)\pi} du \ \cos u$$
$$= \frac{a}{2\pi}\left(\frac{1}{(n-m)}\sin u\Big|_0^{(n-m)\pi} - \frac{1}{n+m}\sin u\Big|_0^{(n+m)\pi}\right) = 0$$

q.e.d.

However the functions are not yet normalized, in fact

$$\int_0^a dx \quad \sin^2\left(\frac{n\pi}{a}x\right) = \frac{a}{n\pi}\int_0^{n\pi} du \; \sin^2 u = \frac{a}{n\pi}\,\frac{1}{2}\,n\pi = \frac{a}{2}$$

for $n \neq 0$

and $\int_0^a dx \; \sin^2\left(0\frac{\pi}{a}x\right) = 0$ for $n = 0$

Therefore an orthonormal set of functions will be

$$f_n(x) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{n\pi}{a}x\right)$$

One could now look for a set of orthonormal functions in the range [-a/2,a/2]
In order to do this one can simply shift the argument x:

$$x \to x' + \frac{a}{2}$$

(drop the prime after the shift)

$$f_n \to \sqrt{\frac{2}{a}}\sin\left[\frac{n\pi}{a}\left(x+\frac{a}{2}\right)\right]$$

$$= \sqrt{\frac{2}{a}}\,\sin\left(\frac{n\pi}{a}x + \frac{n\pi}{2}\right)$$

$$= \sqrt{\frac{2}{a}}\left[\sin\left(\frac{n\pi}{a}x\right)\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\cos\left(\frac{n\pi}{a}x\right)\right]$$

if $n = 2m$ $\qquad \sin m\pi = 0$

$$f_{2m} = \sqrt{\frac{2}{a}}\,\sin\left(\frac{2m\pi}{a}x\right)\underbrace{(-1)^m}$$

can drop this because it is simply a ± 1 factor if $n = 2m - 1$

$$\sin\left(m\pi - \frac{\pi}{2}\right) = \sin(m\pi)\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\cos m\pi$$
$$= (-1)^{m+1}$$

$$f_{2m-1} = \sqrt{\frac{2}{a}}\cos\left(\frac{(2m-1)\pi}{a}x\right) \underbrace{(-1)^{m+1}}_{\text{can drop}}$$

Therefore the set of orthonormal functions in the interval [-a/2,a/2] is

$$f_n(x) = \begin{cases} n=1 & \frac{1}{\sqrt{a}} \\ n \text{ even} & \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right) = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\frac{n}{2}\right) \\ n \text{ odd} & \sqrt{\frac{2}{a}}\cos\left(\frac{(n-1)\pi}{a}x\right) = \sqrt{\frac{2}{a}}\cos\left(\frac{2\pi x}{a}\frac{n-1}{2}\right) \end{cases}$$

One can then observe that

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \qquad \cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

The above should make it clear that another complete set of functions over the interval [-a/2,a/2] is

$$U_m(x) = \frac{1}{\sqrt{a}} e^{i\left(\frac{2\pi x}{a} m\right)} \qquad m = 0, \pm 1, \pm 2, \pm 3 \ldots$$

Problem: Check that the functions are correctly normalized, i.e.

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} dx\, U_m^*(x)\, U_n(x) = \delta_{nm}$$

A generic function f can therefore be written as

$$f(x) = \frac{1}{\sqrt{a}} \sum_{m=-\infty}^{+\infty} A_m \exp\left(\frac{2\pi i x}{a} m\right)$$

$$A_m = \frac{1}{\sqrt{a}} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \, \exp\left(-\frac{2\pi i x}{a} m\right) f(x)$$

At this stage we can take the continuum limit (i.e. send a to infinity)

$$a \to \infty \qquad \frac{2\pi m}{a} \to k \qquad \sum_m \to \frac{a}{2\pi} \int_{-\infty}^{+\infty}$$

$$A_m \to \sqrt{\frac{2\pi}{a}} \, A(k)$$

factor needed to fix normalization

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \, A(k) \, e^{ikx} \\ A(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, f(x) \, e^{-ikx} \end{aligned}}$$

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In taking the limit one traded a numerable parameter m with a continuous parameter k (k can be a quantity with physical dimensions). The orthonormal set of functions parameterized by k is therefore

$$U(k) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

In this case, the orthogonality and completeness relations show complete symmetry in the exchange x <--> k

$$\boxed{\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx \; e^{i(k-k')x} &= \delta(k-k') \\ \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \; e^{ik(x-x')} &= \delta(x-x') \end{aligned}}$$

ORTHOGONALITY

COMPLETNESS