Completeness and orthogonality of functions

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The method of separation of functions allows one to identify classes of functions which are complete and orthogonal on a given interval.

Completeness: A set of functions f_n defined on a given interval is said to be complete if any "reasonably smooth" function g defined on the same interval can be written as a linear combination of the functions f_n

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

For example

$$f_n = \sin\left(\frac{n\pi}{a} x\right)$$

Is a set of complete functions over the interval [0,a]. Typically the mathematical proof that a set of functions is complete is difficult.

Orthogonality: A set of functions f_n is said to be orthogonal over an interval [a,b] if the integral of the product of two different functions of the set over the interval is zero

$$\int_a^b f_n(x) f_m(x)\, dx = 0 \qquad \text{if } n \neq m$$

The sine functions mentioned above are orthogonal over the interval [a,b]

It is in general useful to work with a set of orthonormal functions, by requiring that the integral of the function over the interval [a,b] is normalized to 1

$$f_n \longrightarrow \underbrace{U_n}_{\text{normalized function}} \qquad \underbrace{\int_a^b dx \, |U_n(x)|^2 = 1}_{\text{normalization condition}}$$

We use the absolute value because the function can in general return complex values

$$|U_n(x)|^2 \equiv U_n^*(x) \, U_n(x)$$

The orthogonality and normalization conditions can be put together in a single orthonormality condition

$$\boxed{\int_a^b dx \, U_n^*(x) \, U_m(x) = \delta_{nm}}$$

Let's assume that we want to approximate a function g with a combination of a finite number N of U_n functions:

$$g(x) \simeq \sum_{n=1}^{N} a_n U_n(x)$$

One can then choose the value of the coefficients a_n by minimizing the mean square error defined as

$$M_N = \int_a^b dx \left| g(x) - \sum_{n=1}^{N} a_n U_n(x) \right|^2$$

The coefficients that minimize M_n are

$$\boxed{a_n = \int_a^b U_n^*(x) \, g(x) \, dx}$$

Proof

(suppress arguments, use Einstein's convention for repeated indices)

$$M_N = \int_a^b dx \left(g^* - a_p^* U_p^* \right) \left(g - a_q U_q \right)$$

$$= \int_a^b dx \left(|g|^2 - a_p^* U_p^* g - a_q U_q g^* + a_p^* a_q U_p^* U_q \right)$$

Consider the last integral

$$\int_a^b dx \; a_p^* a_q \; U_p^*(x) \, U_q(x) = a_p^* a_q \underbrace{\int_a^b dx \; U_p^*(x) \, U_q(x)}_{\delta_{pq}}$$

$$= a_p^* a_p$$

Therefore

$$M_N = \int_a^b dx \left(|g(x)|^2 - a_p^* U_p^*(x) \, g(x) - a_q U_q(x) \, g^*(x) \right)$$

$$+ a_p^* a_p$$

$$\frac{\partial M_N}{\partial a_n^*} = 0 = - \int_a^b dx \; U_n^*(x) \, g(x) + a_n$$

$$a_n = \int_a^b dx \; U_n^*(x) \, g(x)$$

q.e.d.

If the set of functions is complete in the sense defined above, the approximation or the function g as a sum of functions U_n improves as N grows. Formally one can say that a set of functions is complete an interval [a,b] if

$$\forall \delta > 0 \quad \exists N_0 \quad \text{such that for} \quad N > N_0 \quad M_N < \delta$$

In this case, as stated above, one can rewrite a function f defined over the interval [a,b] as the sum of a series depending on the complete set of functions U_n

$$f(x) = \sum_{n=1}^{\infty} a_n U_n(x) \qquad a_n = \int_a^b dx\, U_n^*(x) f(x)$$

Now let's replace the second equation above in the first one

$$f(x) = \sum_{n=1}^{\infty} \int_a^b dx'\, U_n^*(x') f(x') U_n(x)$$

$$= \int_a^b dx'\, f(x') \left[\sum_{n=1}^{\infty} U_n^*(x') U_n(x) \right]$$

The equation above is satisfied if the completeness (or closure) relation holds:

$$\boxed{\sum_{n=1}^{\infty} U_n^*(x') U_n(x) = \delta(x - x')}$$

Compare the above with the orthogonality relation

$$\int_a^b dx\, U_n^*(x) U_m(x) = \delta_{nm}$$

In the orthogonality relation one integrates (sums over a continuous variable) x, in the completeness one sums over n. The role of x and n is "interchanged" in the two relations.