Electrostatics-Coulomb's Law

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If we assume to know the sources (charge density and current density), ME allow us to determine the fields B and E, up to a linear combination of plane waves that are not depending from the sources. Here we assume that the only fields that are around are the ones which are generated by the sources.

In addition, we want to start by looking at the electric field generated by a static (i.e. time independent) charge distribution. From the continuity equation we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \bar{J} = 0 \qquad \text{if } \frac{\partial \rho}{\partial t} = 0 \text{ then } \nabla \cdot \bar{J} = 0$$

Static limit for E and B

Observe that if E and B are time independent ME decouple:

$$\boxed{\begin{aligned} \nabla \cdot \bar{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \times \bar{E} &= 0 \end{aligned}} \qquad \boxed{\begin{aligned} \nabla \cdot \bar{B} &= 0 \\ \nabla \times \bar{B} &= \mu_0 \bar{J} \end{aligned}}$$

electrostatics magnetostatics

Electric potential and Poisson's equation

A curl-less vector can always be written as a gradient of a scalar function, which is called the scalar potential (or simply potential when there are no ambiguities).

$$\nabla \times \bar{E} = 0 \longrightarrow \bar{E} = -\nabla \varphi$$

(the minus sign is a convention)

The potential is defined up to an additive constant:

$$\bar{E} = -\nabla \varphi = -\nabla (\varphi + \varphi_0)$$

constant

One can then rewrite Gauss' law in terms of the potential and find the Poisson's equation:

$$\nabla \cdot (-\nabla \varphi) = -\Delta \varphi \longrightarrow \boxed{\Delta \varphi = -\frac{\rho}{\varepsilon_0}}$$ Poisson's Equation

Formal solution of Poisson's equation

Poisson's equation has a useful solution for a charge distribution localized in a finite volume. In this solution, the constant is fixed by requiring that the potential vanishes infinitely far away from the charge

$$\boxed{\varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}}$$

The equation above is indeed a solution of Poisson's equation since

$$\Delta_{\bar{r}}\, \varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \rho(\bar{r}') \underbrace{\Delta_{\bar{r}} \frac{1}{|\bar{r}-\bar{r}'|}}_{-4\pi\,\delta^{(3)}(\bar{r}-\bar{r}')}$$

$$= -\frac{1}{\varepsilon_0} \int d^3r' \, \rho(\bar{r}')\, \delta^{(3)}(\bar{r}-\bar{r}')$$

$$= -\frac{\rho(\bar{r})}{\varepsilon_0} \qquad \text{q.e.d.}$$

Electric field for a localized charge distribution

From the above expression for the potential we can find an expression for E

$$\bar{E} = -\nabla\varphi(\bar{r}) = -\frac{1}{4\pi\varepsilon_0} \int d^3r' \, \rho(\bar{r}') \, \nabla \frac{1}{|\bar{r}-\bar{r}'|} = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3} \rho(\bar{r}')$$

Coulomb's law

Coulombs law is the force applied by one charge distribution on another charge distribution, and it can be obtained starting from the expression for E above

$$\bar{F} = \int d^3r\, \tilde{\rho}(\bar{r})\, \bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r \int d^3r'\, \tilde{\rho}(\bar{r})\, \rho(\bar{r}')\, \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3}$$

If we consider two point like charges we immediately recover the form for Coulomb's law found at the beginning of any elementary book on electromagnetism:

$$\tilde{\rho}(\bar{r}) \equiv Q_1 \delta^{(3)}(\bar{r}-\bar{R}_1) \quad , \quad \rho(\bar{r}') = Q_2\, \delta^{(3)}(\bar{r}'-\bar{R}_2)$$

$$\bar{F} = \frac{1}{4\pi\varepsilon_0} \int d^3r\, Q_1\, \delta^{(3)}(\bar{r}-\bar{R}_1) \int d^3r'\, Q_2\, \delta^{(3)}(\bar{r}'-\bar{R}_2)\, \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{|\bar{R}_1-\bar{R}_2|^3} \underbrace{\bar{R}_1-\bar{R}_2}_{\equiv \bar{d}} = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{d^2}\, \underbrace{\hat{d}}_{\text{unit vector pointing from } \bar{R}_2 \text{ to } \bar{R}_1}$$

By exchanging the two charge densities one finds that

$$\bar{F}' = \frac{1}{4\pi\varepsilon_0} \int d^3r' \int d^3r\, \rho(r)\, \tilde{\rho}(r')\, \frac{\bar{r}'-\bar{r}}{|\bar{r}'-\bar{r}|^3} = -\bar{F}$$

Newton's third law is satisfied.
If the charge densities are identical we have an odd integrand over an even domain, therefore F = 0.

Examples: electric field on the axis of a uniformly charged ring, electric field on the axis of a uniformly charged disk and a uniformly charged plane.

Conservative force

Coulomb's force is conservative: The work done by the electrostatic force on a test charge when the charge moves from A to B depends only on the value of the potential at the points A and B.

$$W = \int_{\bar r_A}^{\bar r_B} d\bar\ell \cdot \bar F = \underbrace{q}_{\text{test charge}} \int_{\bar r_A}^{\bar r_B} d\bar\ell \cdot \bar E = -q \int_{\bar r_A}^{\bar r_B} d\bar\ell \cdot \nabla\varphi$$

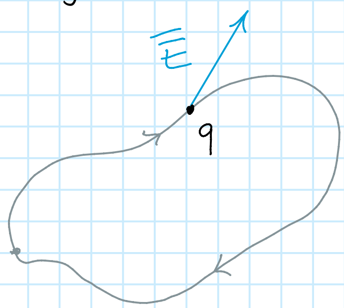
$$= -q \left\{ \int_{\{x_A, y_A, z_A\}}^{\{x_B, y_A, z_A\}} dx \frac{\partial}{\partial x}\varphi + \int_{\{x_B, y_A, z_A\}}^{\{x_B, y_B, z_A\}} dy \frac{\partial}{\partial y}\varphi + \int_{\{x_B, y_B, z_A\}}^{\{x_B, y_B, z_B\}} dz \frac{\partial}{\partial z}\varphi \right\}$$

$$= -q \left\{ \cancel{\varphi(x_B, y_A, z_A)} - \varphi(x_A, y_A, z_A) + \cancel{\varphi(x_B, y_B, z_A)} - \cancel{\varphi(x_B, y_A, z_A)} + \varphi(x_B, y_B, z_B) - \cancel{\varphi(x_B, y_B, z_A)} \right\}$$

$$= q \left[\varphi(x_A, y_A, z_A) - \varphi(x_B, y_B, z_B) \right]$$

The result above is independent from the path chosen (As one can see if one imagines to carry out the integral over any other sequence of points connecting A to B)

By using Stokes' theorem one can directly prove that the work done by the work done by Coulomb's force is zero.



$$W = q \oint d\bar\ell \cdot \bar E = q \int_S d\bar s \; \underbrace{\nabla \times \bar E}_{\substack{=0 \\ \text{in the electrostatic case}}} = 0$$

What was discussed above implies

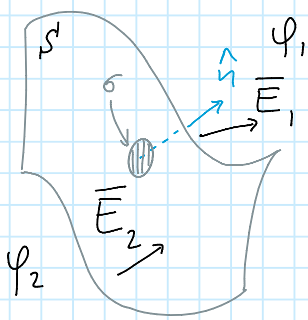
$$\varphi(\bar r) - \varphi(\bar r_0) = -\int_{\bar r_0}^{\bar r} d\bar\ell \cdot \bar E$$

When one set to zero the potential at the initial point of the integration (typically a point infinitely far away from the charge distribution) the expression above simplifies to

$$\varphi(\bar r) = -\int_{\bar r_0}^{\bar r} d\bar\ell \cdot \bar E$$

Matching conditions for the potential

The matching conditions for E on two opposite sides of a surface with a surface charge density were found to be



$$\hat n \cdot \left(\bar E_1 - \bar E_2\right)_{\substack{\text{at}\\ \text{surface}}} = \frac{\sigma}{\varepsilon_0}$$

$$\underbrace{\hat n \cdot \nabla}_{\frac{\partial}{\partial n}} \bar E_2 - \hat n \cdot \nabla \bar E_1 = \frac{\sigma}{\varepsilon_0}$$

$$\boxed{\left(\frac{\partial}{\partial n}\varphi_2 - \frac{\partial}{\partial n}\varphi_1\right)_{\substack{\text{at}\\ \text{surface}}} = \frac{\sigma}{\varepsilon_0}}$$

Notice however that the potential itself is continuous on the surface

$$\left(\varphi_1 - \varphi_2\right)_{\substack{\text{at}\\ \text{surface}}} = 0$$

More precisely one can write

$$\lim_{\varepsilon \to 0} \left(\varphi_1(\bar r_s + \varepsilon \hat n) - \varphi_2(\bar r_s - \varepsilon \hat n)\right) = 0 \qquad \bar r_s \in \text{surface } S$$