Matching conditions for E and B

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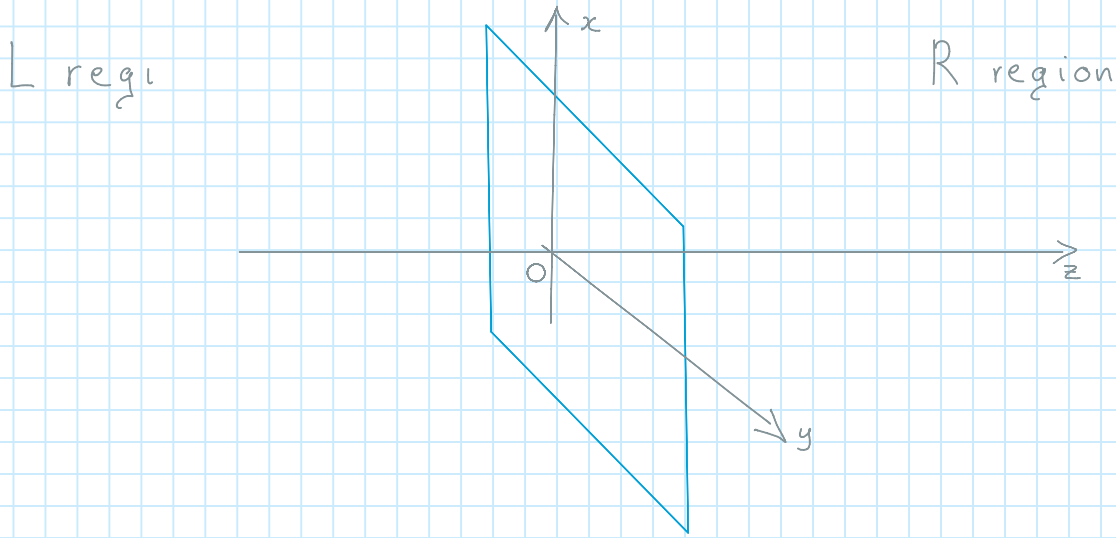
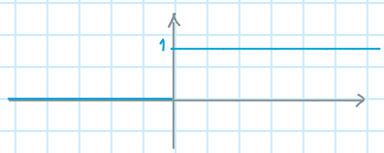
Let's see how the electric field behaves near a surface separating two materials. Consider a flat interface separating the two media located at z = 0



Introduce the Heaviside theta function

$$\vartheta(z) = \begin{cases} 1 & z > 0 \\ 0 & z < 0 \end{cases}$$



It is then possible to write the electric field as follows

$$\bar{E}(\bar{r}) = \bar{E}_L(\bar{r})\,\vartheta(-z) + \bar{E}_R(\bar{r})\,\vartheta(z)$$

We can do a similar thing for the charge density, but one has to allow for the presence of a surface charge density at z = 0

$$\rho(\bar{r}) = \rho_L(\bar{r})\,\vartheta(-z) + \rho_R(\bar{r})\,\vartheta(z) + \sigma(x,y)\,\delta(z)$$

Now we want to apply the divergence operator to E and then, according to Gauss' law, equate the result to the charge density. For this reason we need to use the following mathematical identity

$$\nabla\cdot\left(\bar{f}\,\vartheta(\pm z)\right) = \nabla\cdot\bar{f}\,\vartheta(\pm z) + \bar{f}\cdot\nabla\vartheta(\pm z) = \nabla\cdot\bar{f}\,\vartheta(\pm z) \pm \delta(z)\,\bar{f}\cdot\hat{k}$$

Therefore

$$\nabla\cdot\bar{E} = \nabla\cdot\bar{E}_L\,\vartheta(-z) - \delta(z)\,\bar{E}_L\cdot\hat{k} + \nabla\cdot\bar{E}_R\,\vartheta(z) + \delta(z)\,\bar{E}_R\cdot\hat{k}$$

$$= \nabla\cdot\bar{E}_L\,\vartheta(-z) + \nabla\cdot\bar{E}_R\,\vartheta(z) + \delta(z)\,\hat{k}\cdot\left(\bar{E}_R - \bar{E}_L\right)$$

However, Gauss' law is valid separately in the L and R regions

$$\nabla\cdot\bar{E}_L = \frac{\rho_L}{\varepsilon_0} \quad , \quad \nabla\cdot\bar{E}_R = \frac{\rho_R}{\varepsilon_0}$$

So that

$$\nabla\cdot\bar{E} = \frac{\rho_L}{\varepsilon_0}\vartheta(-z) + \frac{\rho_R}{\varepsilon_0}\vartheta(+z) + \delta(z)\,\hat{k}\cdot\left(\bar{E}_R - \bar{E}_L\right)$$

$$= \frac{1}{\varepsilon_0}\left(\rho_L\vartheta(-z) + \rho_R\vartheta(+z) + \sigma\;\delta(z)\right)$$

$$\boxed{\hat{k}\cdot\left[\bar{E}_R - \bar{E}_L\right]_{z=0} = \frac{\sigma}{\varepsilon_0}}$$

The component of the electric field perpendicular to the interface is discontinuous.

Now we want to apply the same procedure to the magnetic field B, but the right hand side of Gauss' equation for the magnetic field is zero, so that we immediately conclude that the component of the magnetic field which is perpendicular to the surface is continuous

$$\boxed{\hat{k}\cdot\left[\bar{B}_R - \bar{B}_L\right]_{z=0} = 0}$$

We now consider Faraday's law and we take the curl of the electric field

$$\nabla\times\left(\bar{f}\,\vartheta(\pm z)\right) = \vartheta(\pm z)\;\nabla\times\bar{f} - \bar{f}\times\nabla\vartheta(\pm z)$$

$$= \vartheta(\pm z)\,\nabla\times\bar{f} \mp \bar{f}\times\hat{k}\;\;\delta(z)$$

$$= \vartheta(\pm z)\,\nabla\times\bar{f} \pm \hat{k}\times\bar{f}\;\;\delta(z)$$

Consequently

$$\nabla \times \bar{E} = \vartheta(-z)\, \nabla \times \bar{E}_L + \vartheta(+z)\, \nabla \times \bar{E}_R + \hat{k} \times \left[\bar{E}_R - \bar{E}_L \right] \delta(z)$$

But also

$$-\frac{\partial \bar{B}}{\partial t} = -\vartheta(-z) \frac{\partial \bar{B}_L}{\partial t} - \vartheta(+z) \frac{\partial \bar{B}_R}{\partial t}$$

By using then

$$\nabla \times \bar{E}_L = -\frac{\partial \bar{B}_L}{\partial t} \quad , \quad \nabla \times \bar{E}_R = -\frac{\partial \bar{B}_R}{\partial t}$$

One finds, that the components of the electric field which are parallel to the interface are continuous

$$\boxed{\hat{k} \times \left[\bar{E}_R - \bar{E}_L \right]_{z=0} = 0}$$

Finally we take the curl of B to see what are the consequences of Ampere-Maxwell's equation near the surface that separates the two media

$$\nabla \times \bar{B} = \vartheta(-z)\, \nabla \times \bar{B}_L + \vartheta(z)\, \nabla \times \bar{B}_R + \hat{k} \times \left[\bar{B}_R - \bar{B}_L \right] \delta(z)$$

$$-\frac{1}{c^2} \frac{\partial \bar{E}}{\partial t} = \vartheta(-z) \left[-\frac{1}{c^2} \frac{\partial \bar{E}_L}{\partial t} \right] + \vartheta(+z) \left[-\frac{1}{c^2} \frac{\partial \bar{E}_R}{\partial t} \right]$$

$$\mu_0 \bar{J}(\bar{r}) = \mu_0 \bar{J}_L(\bar{r})\, \vartheta(-z) + \mu_0 \bar{J}_R(\bar{r})\, \vartheta(+z) + \underbrace{\mu_0 \bar{K}(x,y)}_{\text{surface current density}} \delta(z)$$

By using

$$\nabla \cdot \bar{B}_L - \frac{1}{c^2} \frac{\partial \bar{E}_L}{\partial t} = \mu_0 \bar{J}_L \quad , \quad \nabla \cdot \bar{B}_R - \frac{1}{c^2} \frac{\partial \bar{E}_R}{\partial t} = \mu_0 \bar{J}_R$$

One finds

$$\boxed{\hat{k} \times \left[\bar{B}_R - \bar{B}_L \right]_{z=0} = \mu_0 \bar{K}}$$

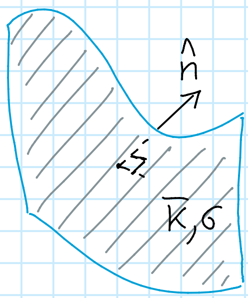
The components of B parallel to the surface are discontinuous.

These four boundary conditions are local, and therefore are valid for surfaces of any shape.

Summary table

fields $\bar{B}_1, \bar{E}_1$

fields $\bar{B}_2, \bar{E}_2$



$$\hat{n} \cdot [\bar{E}_1 - \bar{E}_2] = \frac{\sigma}{\varepsilon_0}, \quad \hat{n} \cdot [\bar{B}_1 - \bar{B}_2] = 0$$

$$\hat{n} \times [\bar{E}_1 - \bar{E}_2] = 0, \quad \hat{n} \times [\bar{B}_1 - \bar{B}_2] = \mu_0 \bar{K}$$