Maxwell's equations in vacuum

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All what one needs to know about electromagnetism is included in Maxwell's equations

non-homogeneous equations

$$\nabla \cdot \overline{E} = \frac{\rho}{\varepsilon_0}$$ GAUSS LAW FOR $\overline{E}$

$$\nabla \times \overline{B} - \frac{1}{c^2}\frac{\partial \overline{E}}{\partial t} = \mu_0 \overline{J}$$ AMPERE - MAXWELL LAW

homogeneous equations

$$\nabla \cdot \overline{B} = 0$$ GAUSS LAW FOR $\overline{B}$

$$\nabla \times \overline{E} + \frac{\partial \overline{B}}{\partial t} = 0$$ FARADAY INDUCTION LAW

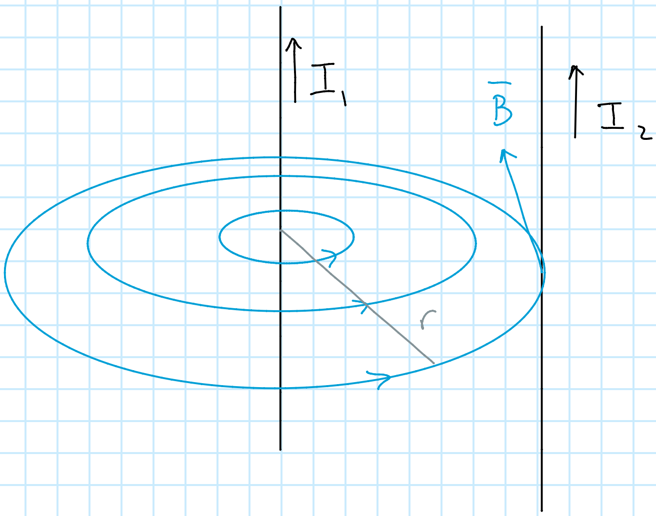
$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 299\,792\,458 \ \frac{m}{s}$$ SPEED OF LIGHT IN VACUUM

$$\mu_0 = 4\pi \times 10^{-7} \ \frac{V\,s}{A\,m}$$ MAGNETIC PERMEABILITY OF VACUUM

$$\varepsilon_0 = 8.854\ldots \times 10^{-12} \ \frac{C}{V\,m} \ \left(\frac{A\,s}{V\,m}\right)$$ ABSOLUTE DIELECTRIC PERMITTIVITY OF VACUUM

Maxwell originally wrote twelve equations involving fields. The synthesis to four independent equations is due to Heaviside and Hertz.

The definition of μ_o follows from the definition of Ampere as the unit to measure current: The force per unit of length between two straight wires carrying a current of 1 A, parallel to each other and held at a distance of 1 meter is 2×10^{-7} N/m



AMPERE - MAXWELL'S LAW

$$2\pi r B = \mu_o I_1 \qquad B = \frac{\mu_o}{2\pi} \frac{I_1}{r}$$

$$\overline{F} = I_2 \, \overline{\ell} \times \overline{B}$$

$$F = \frac{\mu_o}{2\pi} \frac{I_1 I_2}{r} \ell$$

$$\frac{dF}{d\ell} = \frac{\mu_o}{2\pi} \frac{I_1 I_2}{r} = \frac{\mu_o}{2\pi} \frac{1A \; 1A}{1m} \equiv 2 \times 10^{-7} \frac{N}{m}$$

$$\hookrightarrow \quad \mu_o = 4\pi \times 10^{-7} \frac{N}{A^2} = 4\pi \times 10^{-7} \frac{\frac{V}{m}\cancel{C}}{\frac{\cancel{C}}{s} A} = \frac{Vs}{Am}$$

$$\boxed{\text{Work done by electromagnetic fields on a charge}}$$

$$dW = \overline{F} \cdot d\overline{r} = \overline{F} \cdot \frac{d\overline{r}}{dt} dt = \overline{F} \cdot \overline{v} \, dt$$

$$= q(\overline{E} + \overline{v} \times \overline{B}) \cdot \overline{v} \, dt = q \, \overline{E} \cdot \overline{v} \, dt$$

since $\overline{v} \times \overline{B}$ is $\perp$ to both $\overline{v}$ and $\overline{B}$ and therefore $(\overline{v} \times \overline{B}) \cdot \overline{v} = 0$

The B field does not do work on the charge

Structure of Maxwell's equations

Maxwell's equations are partial differential equations for E and B (taking current and charge densities as given). ME deal with fields, the connection to forces and measurable effects is provided by the Coulomb-Lorentz equation. For continuous charge and current densities the equation can be written as

$$\bar{F}(t) = \int d^3r \left[\hat{\rho}(\bar{r},t)\,\bar{E}(\bar{r},t) + \hat{\bar{J}}(\bar{r},t) \times \bar{B}(\bar{r},t) \right]$$

We use the hat notation to indicate that the charge and current densities are not the ones that generate the fields E and B in the equation.

ME are

1) First order differential equations, they involve only first derivatives with respect to x,y,z, and t.
2) Linear equations, only first powers of E and B appear in the equations.
3) Inhomogeneous equations, the charge and current densities are source terms that do not depend on the fields.

In ME as written above ρ and $\bar{J}$ represent the complete contributions to the charge and current densities. Consider for example a charge density ρ_I placed near a dielectric. The charge density will create a field E which will polarize the dielectric and create a second charge density in the dielectric due to the polarization which we indicate with ρ_D. The complete charge density which appears in ME will be the sum of the two densities mentioned above

$$\rho = \rho_I + \rho_D$$

At this stage we will not attempt to calculate ρ_D

Solutions of Maxwell's equations

Does a solution of ME exist for every charge and current density? Take the time derivative of the equation involving ρ and the divergence of the equation involving the current density $\bar{J}$

$$\frac{\partial}{\partial t}\left(\nabla\cdot\bar{E}\right) = \nabla\cdot\frac{\partial}{\partial t}\bar{E} = \frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial t} \qquad (a)$$

$$\underbrace{\nabla\cdot(\nabla\times\bar{B})}_{=0} - \frac{1}{c^2}\nabla\cdot\frac{\partial\bar{E}}{\partial t} = \mu_0\nabla\cdot\bar{J}$$

$$\hookrightarrow \qquad \nabla\cdot\frac{\partial\bar{E}}{\partial t} = -c^2\mu_0\nabla\cdot\bar{J} \qquad (b)$$

$$(a) = (b) \longrightarrow \frac{\partial\rho}{\partial t} = -c^2\varepsilon_0\mu_0\nabla\cdot\bar{J} = -\nabla\cdot\bar{J}$$

$$\hookrightarrow \quad \frac{\partial\rho}{\partial t} + \nabla\cdot\bar{J} = 0$$ Continuity equation

Notice that the presence of Maxwell's displacement current $\varepsilon_0\frac{\partial\bar{E}}{\partial t}$ is crucial in the derivation of the continuity equation above. Without that term ME are not consistent with the continuity equation.

Another way of looking at Maxwell's equations

Consider the two equations involving time derivatives

$$\frac{\partial\bar{E}}{\partial t} = c^2\nabla\times\bar{B} - \frac{1}{\varepsilon_0}\bar{J} \quad ; \quad \frac{\partial\bar{B}}{\partial t} = -\nabla\times\bar{E}$$

Given $\bar{J}(\bar{r},t)$ for $t>t_0$ and $\bar{E}(\bar{r},t_0)$, $\bar{B}(\bar{r},t_0)$ E and B are determined at all times $t>t_0$

Observe: The field E and B at t_0 are not arbitrary, they have to satisfy

$$\nabla\cdot\bar{E} = \frac{\rho}{\varepsilon_0} \quad ; \quad \nabla\cdot\bar{B} = 0$$

The two equations above are constraints that have to be satisfied by E and B at all times.

The two equations involving time derivatives fix the time evolution of E and B in such a way that the equations without time derivatives are satisfied the whole time:

$$\nabla \cdot \frac{\partial \bar{E}}{\partial t} = \nabla \cdot \left(c^2 \nabla \times \bar{B} - \frac{1}{\varepsilon_0} \bar{J} \right) = -\frac{1}{\varepsilon_0} \nabla \cdot \bar{J} = \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial t}$$

$$\hookrightarrow \quad \frac{\partial}{\partial t} \left(\nabla \cdot \bar{E} - \frac{\rho}{\varepsilon_0} \right) = 0$$

if $\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0}$ at $t = t_0$ then $\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \quad \forall t$

Similarly

$$\nabla \cdot \frac{\partial \bar{B}}{\partial t} = -\nabla \cdot (\nabla \times \bar{E}) = 0 \longrightarrow \frac{\partial}{\partial t} (\nabla \cdot \bar{B}) = 0$$