Point charge and a grounded conducting sphere

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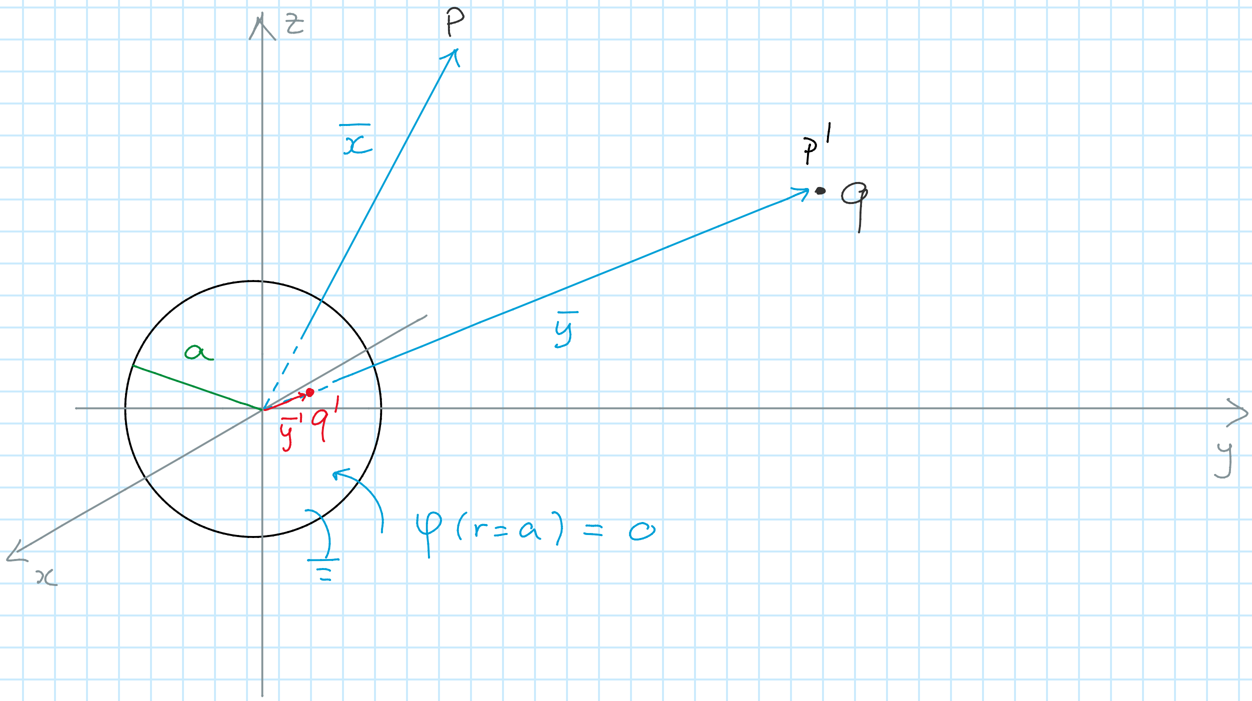
One can use the method of images to find the potential in the case of a charge q placed in the space surrounding a grounded conducting sphere



The radius of the sphere is indicated with the letter a. The problem can be solved with the method of images. The symmetry of the problem dictates that the image charge should be placed on the line OP'. One needs to determine the magnitude of the image charge and the distance where this charge should be placed from the origin. The potential due to the point charge and to the image charge will be

$$\varphi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\bar{x}-\bar{y}|} + \frac{q'}{|\bar{x}-\bar{y}'|}\right)$$

Now one needs to impose that the potential is zero on the surface of the sphere

$$\varphi(|\bar{x}| = a) = 0$$

One can then rewrite

$$\bar{x} = x\,\hat{n} \qquad \bar{y} = y\,\hat{n}' \qquad \bar{y}' = y'\,\hat{n}'$$

same direction

$$\varphi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{x\left|\hat{n} - \frac{y}{x}\hat{n}'\right|} + \frac{q'}{y'\left|\hat{n}' - \frac{x}{y'}\hat{n}\right|}\right)$$

$$\varphi(|\bar{x}| = a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{a\left|\hat{n} - \frac{y}{a}\hat{n}'\right|} + \frac{q'}{y'\left|\hat{n}' - \frac{a}{y'}\hat{n}\right|}\right) = 0$$

$$\left|\hat{n} - \frac{y}{a}\hat{n}'\right|^2 = 1 + \frac{y^2}{a^2} - 2\frac{y}{a}\,\hat{n}\cdot\hat{n}' \qquad (\mathrm{I})$$

$$\left|\hat{n}' - \frac{a}{y'}\hat{n}\right|^2 = 1 + \frac{a^2}{(y')^2} - 2\frac{a}{y'}\,\hat{n}\cdot\hat{n}' \qquad (\mathrm{II})$$

(I) and (II) are the same if

$$\frac{y}{a} = \frac{a}{y'} \implies \boxed{y' = \frac{a^2}{y}}$$

The potential on the surface of the sphere will be zero if, in addition one chooses

$$\frac{q}{a} = -\frac{q'}{y'}$$

$$q' = -\frac{q}{a}y' = -\frac{q}{\cancel{a}}\frac{a^{\cancel{2}}}{y} = -\frac{qa}{y} \qquad \boxed{q' = -\frac{qa}{y}}$$

At the beginning of this discussion we implicitly assumed that $y > a$. The relations above imply then that $y' < a$ and $q' < q$.

$$\text{if } y \to a \quad (\text{from } y > a) \quad \text{then } y' \to a \quad (y' < a)$$

$$\text{and} \quad q' \to -q$$

It is also interesting to calculate the surface charge density. To do this one needs to remember that the electric field inside the conductor is 0 and just outside the conductor one has a field E perpendicular to the conductor's surface. The magnitude of the field is

$$E = \frac{\sigma}{\varepsilon_0} \qquad -\left.\frac{\partial \varphi}{\partial x}\right|_{x \to a^+} = \frac{\sigma}{\varepsilon_0}$$

$x = |\bar{x}|$, not the x coordinate of a point

$$\sigma = -\varepsilon_0 \left.\frac{\partial \varphi}{\partial x}\right|_{x = a^+}$$

It is convenient to calculate the derivative in stages

$$4\pi\varepsilon_0 \varphi(\bar{x}) \equiv \frac{q}{\sqrt{D_1}} - \underbrace{\frac{q}{y} a}_{q'} \frac{1}{\sqrt{D_2}}$$

$$D_1 = x^2 + y^2 - 2xy \underbrace{\cos\gamma}_{\hat{n}\cdot\hat{n}'}$$

$$D_2 = (y')^2 + x^2 - 2xy' \cos\gamma$$

$$4\pi\varepsilon_0 \frac{\partial \varphi}{\partial x} = -\frac{q}{2} \frac{1}{D_1^{\frac{3}{2}}} \frac{\partial D_1}{\partial x} + \frac{qa}{2y} \frac{1}{D_2^{\frac{3}{2}}} \frac{\partial D_2}{\partial x}$$

$$\frac{\partial D_1}{\partial x} = 2x - 2y\cos\gamma = 2(x - y\cos\gamma)$$

$$\frac{\partial D_2}{\partial x} = 2(x - y'\cos\gamma) = 2\left(x - \frac{a^2}{y}\cos\gamma\right)$$

$$D_1\big|_{x=a} = a^2 + y^2 - 2ay\cos\gamma = y^2 \overbrace{\left(1 + \frac{a^2}{y^2} - 2\frac{a}{y}\cos\gamma\right)}^{\equiv D}$$

$$D_2\big|_{x=a} = \frac{a^4}{y^2} + a^2 - 2\frac{a^3}{y}\cos\gamma = a^2 \overbrace{\left(1 + \frac{a^2}{y^2} - 2\frac{a}{y}\cos\gamma\right)}^{= D}$$

Therefore

$$4\pi\varepsilon_0 \left.\frac{\partial\varphi}{\partial x}\right|_{x=a} = -\frac{q}{y^3 D^{\frac{3}{2}}}(a - y\cos\gamma) + \frac{qa}{y}\frac{1}{a^3 D^{\frac{3}{2}}}\left(a - \frac{a^2}{y}\cos\gamma\right)$$

$$= \frac{q}{D^{\frac{3}{2}}}\left\{-\frac{a}{y^3} + \cancel{\frac{1}{y^2}\cos\gamma} + \frac{a}{y}\left(\frac{1}{a^2} - \cancel{\frac{1}{ay}\cos\gamma}\right)\right\}$$

$$= \frac{q}{D^{\frac{3}{2}} a^2}\left(-\frac{a^3}{y^3} + \frac{a}{y}\right) = \frac{q}{D^{\frac{3}{2}}}\frac{1}{ay}\left(1 - \frac{a^2}{y^2}\right)$$

$$\sigma = \varepsilon_0 E = -\varepsilon_0 \left.\frac{\partial\varphi}{\partial x}\right|_{x=a}$$

$$\boxed{\sigma = -\frac{q}{4\pi a y}\frac{\left(1 - \frac{a^2}{y^2}\right)}{\left(1 + \frac{a^2}{y^2} - 2\frac{a}{y}\cos\gamma\right)^{\frac{3}{2}}}}$$

From the equation above one sees that the charge density is largest in the direction facing the point charge q (as expected).

The force applied by the sphere on q is the same as the force applied by the image charge on q

$$\bar{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}\hat{r} \qquad \bar{r} \equiv (y - y')\hat{n}'$$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q|q'|}{(y-y')^2} = \frac{1}{4\pi\varepsilon_0}\frac{q^2 a}{y}\frac{1}{\left(y - \frac{a^2}{y}\right)^2}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q^2 a}{y^3} \frac{1}{\left(1 - \frac{a^2}{y^2}\right)^2}$$

One can calculate the total induced charge by integrating sigma over the sphere

The same kind of discussion applies to the case in which y<a. In that case, the total induced charge is q.

Finally, let's make contact between the general notation used here (borrowed from Jackson) and the notation employed in the notes where the method of images was introduced

$$\bar{x} = \{x, y, z\} \qquad \bar{y} = \{d, 0, 0\} \qquad \begin{array}{l} a \to R \\ \hat{n}' \to \hat{i} \end{array}$$

$$\varphi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\bar{x} - \bar{y}|} - \frac{q\,a}{y\left|\bar{x} - \frac{a^2}{y}\hat{n}'\right|}\right)$$

notation change

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{(x-d)^2 + y^2 + z^2}} - \frac{q\,R}{d\sqrt{\left(x - \frac{R^2}{d}\right)^2 + y^2 + z^2}}\right)$$