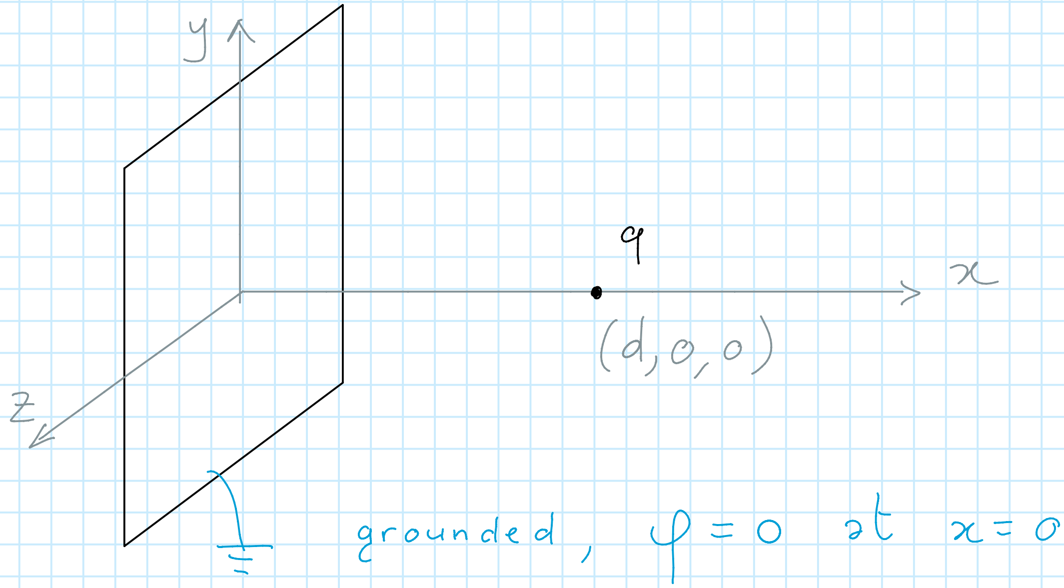
Method of images

Sunday, December 30, 2018 3:28 AM

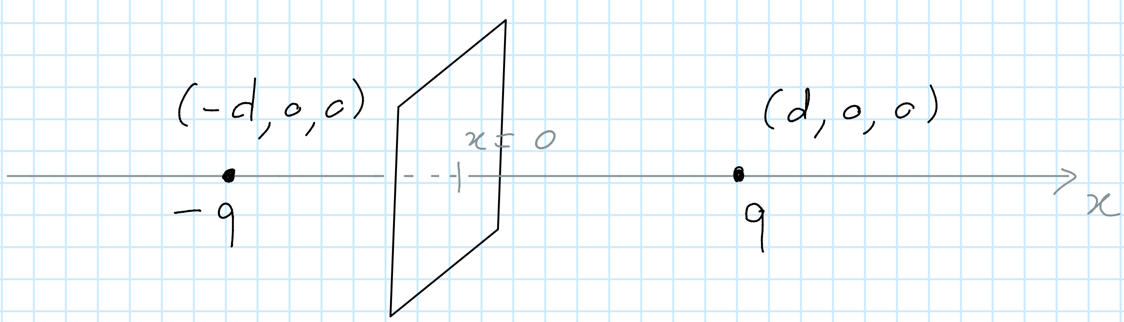
In few simple situations, boundary value problems can be solved by mapping the problem in another problem, in which certain boundary conditions are obtained by considering certain combinations of point charges that are not actually present in the problem. These charges are called image charges or images. Let's consider first the simplest of these situations.

A point-like charge and a conducting plane

Consider a point charge placed in front of a conducting plane



We are interested in finding the potential in the region $x > 0$. This can be done by forgetting about the conductor and imagining that in addition to the charge q placed at $(d,0,0)$ there is a second charge $-q$ placed at the point of coordinates $(-d,0,0)$. This second charge is the image charge for this problem.



In a generic point in space, the electric potential due to the presence of q and -q is

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left\{ \frac{q}{\left[(x-d)^2+y^2+z^2\right]^{\frac{1}{2}}} - \frac{q}{\left[(x+d)^2+y^2+z^2\right]^{\frac{1}{2}}} \right\}$$

It is easy to check that at x = 0 and for arbitrary values of y and z, the potential vanishes. This is exactly the boundary condition we need to consider in the case of a charge q and a grounded conducting plane at x = 0. Consequently, the potential above, due to the two charges q and - q, is also the potential (for x > 0) for the case of a grounded conducting plane at x = 0 and the charge q. Therefore, our initial boundary value problem was solved.

If the region x < 0 is filled by the conductor, the potential in that region is simply zero. The x component of the electric field in the region x > 0 is given by

$$E_x = -\frac{\partial\varphi}{\partial x} = +\frac{q}{4\pi\varepsilon_0}\left\{ \frac{x-d}{\left[(x-d)^2+y^2+z^2\right]^{\frac{3}{2}}} - \frac{x+d}{\left[(x+d)^2+y^2+z^2\right]^{\frac{3}{2}}} \right\}$$

Since E = 0 in the conductor and E is parallel to x just outside the conductor, one can use the discontinuity of E across the conductor's surface in order to find the surface charge density:

$$\sigma = \varepsilon_0 E_x \Big|_{x=0} = -\frac{q}{2\pi}\,\frac{d}{\left(d^2+y^2+z^2\right)^{\frac{3}{2}}}$$

The surface charge is largest at y = z = 0, which not surprisingly is the conductor' point closest to the charge q. One can compute the total charge on the conductor by integrating the surface charge

$$q_{induced} = \int dx\,dy\;\sigma = -\frac{qd}{2\pi}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \;\frac{1}{\left(d^2+x^2+y^2\right)^{\frac{3}{2}}}$$

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy = \int_0^{\infty} dr\, r \int_0^{2\pi} d\phi$$

$$q_{induced} = -qd \int_0^{\infty} dr \frac{r}{(d^2+r^2)^{\frac{3}{2}}} =$$

$$= -\frac{qd}{2} \int_0^{\infty} du \frac{1}{(d^2+u)^{\frac{3}{2}}} = -\frac{qd}{2} \left(-2\right) \frac{1}{(d^2+u)^{\frac{1}{2}}} \Bigg|_0^{\infty}$$

$$= -q$$

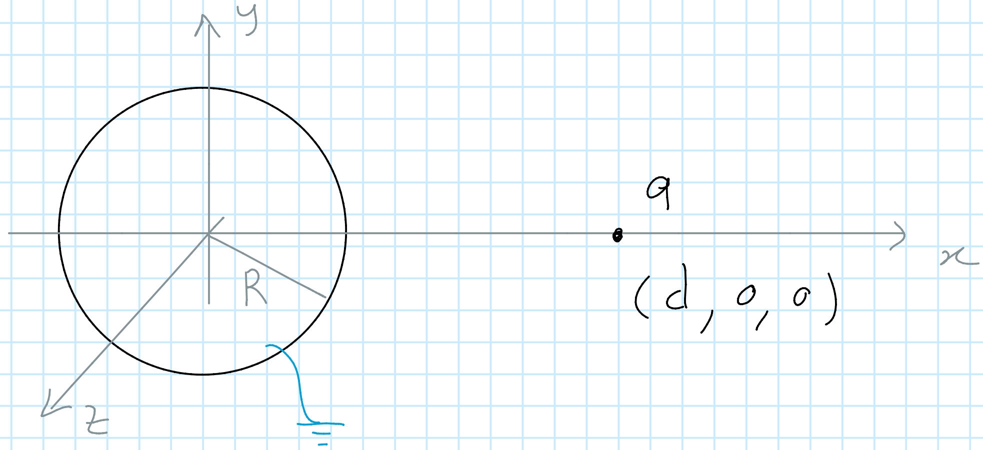
The charge on the conductor is equal to the image charge. This is a general feature of the method of images.

One can also use the method of images to calculate the force that the conductor is applying to the charge q. The force is attractive.

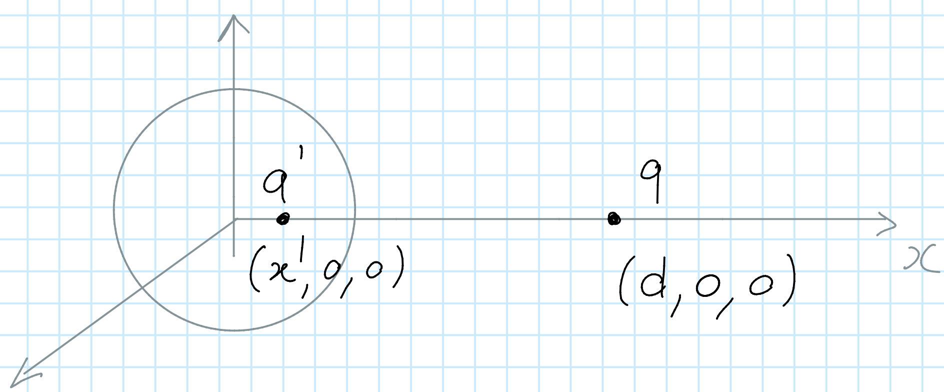
$$\bar{F} = \frac{q^2}{4\pi\varepsilon_0} \frac{1}{(2d)^2} \hat{\imath} = -\frac{1}{16\pi\varepsilon_0} \frac{q^2}{d^2} \hat{\imath}$$

Charged particle near a conducting sphere

The potential due to a point like charge placed in front of a grounded conducting sphere can also be solved with the method of images



The potential outside the sphere is the same that one would have in the following situation



$$q' = -\frac{qR}{d} \qquad x' = \frac{R^2}{d}$$

The values of q' and x' above can be determined by using the method discussed in Jackson (see Jackson notes for details). The potential will then be

$$\varphi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left[(x-d)^2+y^2+z^2\right]^{\frac{1}{2}}} - \frac{R}{d}\,\frac{1}{\left[(x-R^2/d)^2+y^2+z^2\right]^{\frac{1}{2}}}\right)$$

One can check that on the surface of the sphere the potential is indeed zero.

if $x^2+y^2+z^2=R^2$

$$(x-d)^2+y^2+z^2 = x^2+y^2+z^2-2xd+d^2 = R^2-2xd+d^2$$

$$\left(x-\frac{R^2}{d}\right)^2+y^2+z^2 = R^2-2x\frac{R^2}{d}+\frac{R^4}{d^2}$$

$$\varphi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(R^2-2xd+d^2)^{\frac{1}{2}}} - \frac{R}{d}\,\frac{1}{\left(R^2-2x\frac{R^2}{d}+\frac{R^4}{d^2}\right)^{\frac{1}{2}}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[(R^2-2xd+d^2)^{-\frac{1}{2}} - (d^2-2xd+R^2)^{-\frac{1}{2}}\right] = 0$$

One can then find the surface charge and check that the integrated surface charge is equal to q' (see Jackson notes).

One can also consider the case in which the conductor is not grounded but carries a fixed charge Q'. This case can be treated by adding to the grounded sphere case an additional image charge Q – q' at the center of the sphere. This second image charge adds a second radial electric field to the field due to the grounded sphere case. The sum of the two fields is the total electric field for the case in which the sphere is not grounded but isolated (i.e. It carries a fixed overall charge Q).