Symmetric problems and Gauss' law

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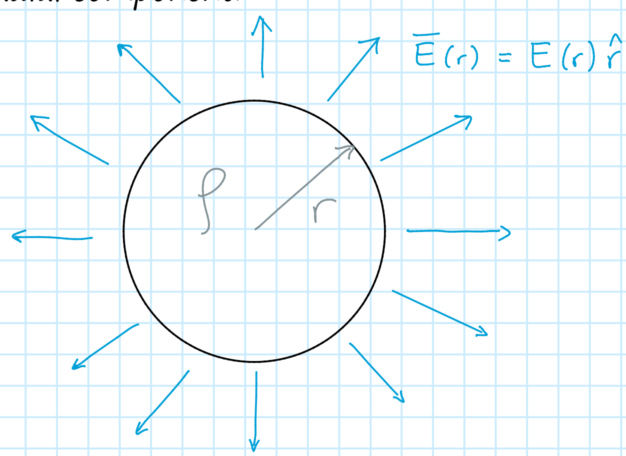
In special symmetric situations it is possible to calculate the electric field using only symmetry and Gauss' law (in integral form)

Uniformly charged sphere

Consider a charged sphere of radius R with uniform charge. Because of the spherical symmetry, the electric field can only have a radial component.



We apply Gauss' theorem to a gaussian surface centered in the center of the sphere

$$\int_V d^3r \; \nabla\cdot\bar{E} = \int_{\partial S} d\bar{s}\cdot\bar{E} = E(\bar{r}) \int_{\partial S} d\bar{s}\cdot\hat{r} = E(r)\, 4\pi r^2 \qquad (a)$$

$$\frac{1}{\varepsilon_0}\int_V d^3r \; \rho = \frac{\rho}{\varepsilon_0}\int_{\substack{\text{sphere}\\ \min\{r,R\}}} d^3r = \begin{cases} \dfrac{\rho}{\varepsilon_0}\dfrac{4}{3}\pi r^3 & r < R \\[2ex] \dfrac{\rho}{\varepsilon_0}\dfrac{4}{3}\pi R^3 & \end{cases} \qquad (b)$$

Therefore, since (a) = (b)

$$E(r) = \begin{cases} \dfrac{\rho}{3\varepsilon_0} r & r < R \\[2ex] \dfrac{\rho}{3\varepsilon_0}\dfrac{R^3}{r^2} & r > R \end{cases}$$

Or, in terms of the total charge

$$Q = \frac{4}{3}\pi R^3 \rho$$

$$E(r) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0}\dfrac{r}{R^3} & r < R \\[2ex] \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{r^2} & r > R \end{cases}$$

same as a point like charge

The corresponding potential is

$$-\int_{\bar{r}_A}^{\bar{r}_B} \bar{E}\cdot d\bar{\ell} = \int_{\bar{r}_A}^{\bar{r}_B} \nabla\varphi \cdot d\bar{\ell} = \varphi(\bar{r}_B) - \varphi(\bar{r}_A)$$

Choose the two points along the same radial direction $\quad R < r_A < r_B$

$$\varphi(r_B) - \varphi(r_A) = -\int_{r_A}^{r_B} E(r)\,dr = -\frac{Q}{4\pi\varepsilon_0}\int_{r_A}^{r_B}\frac{1}{r^2}\,dr = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\Bigg|_{r_A}^{r_B}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

Now send r_B to infinity

$$\cancelto{0}{\varphi(\infty)} - \varphi(r_A) = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r_A}\right) \longrightarrow \varphi(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r} \qquad (r > R)$$

Now choose $r_A < R < r_B$

$$\varphi(r_B) - \varphi(r_A) = -\int_{r_A}^{r_B} E(r)\, dr = -\frac{Q}{4\pi\varepsilon_0}\left\{ \int_{r_A}^{R} dr \frac{r}{R^3} + \int_{R}^{r_B} \frac{dr}{r^2} \right\}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left\{ \frac{1}{2R} - \frac{r_A^2}{2R^3} - \frac{1}{r_B} + \frac{1}{R} \right\}$$

Take the limit for r_B to infinity

$$-\varphi(r_A) = -\frac{Q}{4\pi\varepsilon_0}\frac{1}{R}\left\{ \frac{3}{2} - \frac{r_A^2}{2R^2} \right\}$$

$$\varphi(r) = \frac{Q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) \qquad (r < R)$$

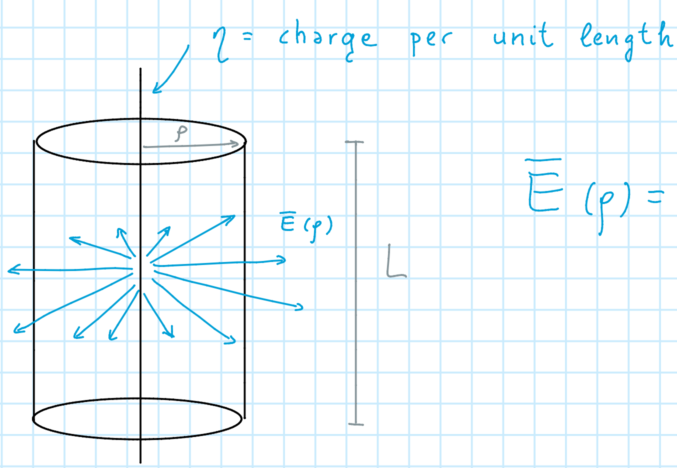
Notice that

$$\lim_{r \to R} \frac{Q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) = \lim_{r \to R} \frac{Q}{4\pi\varepsilon_0 r} = \frac{Q}{4\pi\varepsilon_0 R}$$

The potential is continuous as expected

Long straight wire carrying a uniform charge density

In determining the electric field due to a long straight wire carrying a uniform charge one can exploit the cylindrical symmetry of the problem, choosing a cylindrical gaussian surface with the wire along its axis.



$$\bar{E}(\rho) = E(\rho)\,\hat{\rho}$$

$$\int_{\partial V} \bar{E} \cdot d\bar{s} = \frac{\eta L}{\varepsilon_0}$$

$$E(\rho)\, 2\pi \rho L = \frac{\eta L}{\varepsilon_0} \qquad E(\rho) = \frac{1}{2\pi\varepsilon_0} \frac{\eta}{\rho}$$

The corresponding potential will be

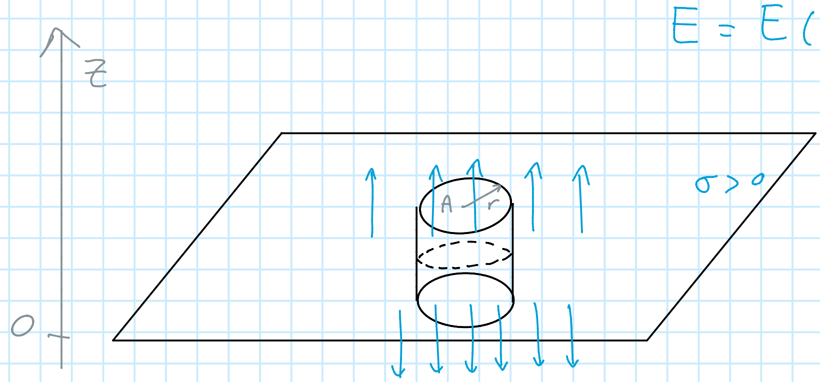
$$\varphi(\bar{r}_A) - \varphi(\bar{r}_B) = -\int_{\bar{r}_A}^{\bar{r}_B} d\bar{\ell} \cdot \bar{E} = -\int_{\rho_A}^{\rho_B} d\rho \frac{1}{2\pi\varepsilon_0} \frac{\eta}{\rho} = \frac{\eta}{2\pi\varepsilon_0} \ln \rho \Big|_{\rho_A}^{\rho_B} = \frac{\eta}{2\pi\varepsilon_0} \ln \frac{\rho_B}{\rho_A}$$

Where we integrated along a radial path from r_A to r_B. In this case we do not have the possibility the potential at infinity equal to zero (remember that we assumed the wire to be infinitely long, therefore it "looks the same" no matter how far from it the observer is).

Infinite plane, uniformly charged

We now consider a plane infinite surface carrying a constant charge density per unit area. We choose a cylindrical gaussian surface with the flat sides of the cylinder parallel to the surface.

$$\bar{E} = E(z)\,\hat{k}$$



In this case the symmetry of the problems requires that E depends only on z and that it is parallel to the z axis. Consequently, there is no electric field flux through the curved surface of the cylinder. Furthermore E above and below the surface will point in opposite directions.

$$\int_{\partial V} \bar{E} \cdot d\bar{s} = \frac{\sigma A}{\varepsilon_0} \qquad\qquad 2\cancel{A}\,E(z) = \frac{\sigma \cancel{A}}{\varepsilon_0}$$

$$E(z) = \frac{\sigma}{2\varepsilon_0}$$

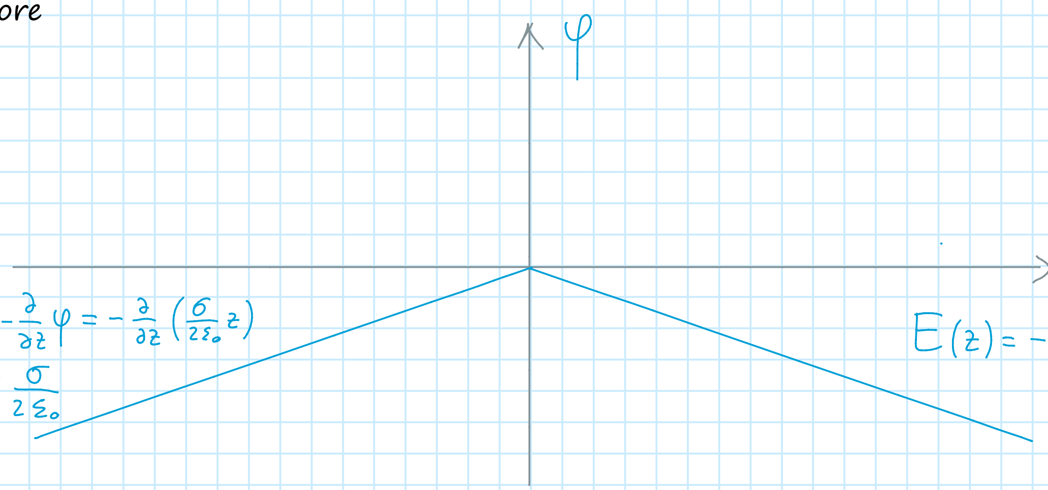
The magnitude of E is indeed independent from z.
The electric field across the plate is discontinuous, as expected (remember the boundary conditions across a charged surface).

$$E(z \to 0^+) - E(z \to 0^-) = \frac{\sigma}{\varepsilon_0}$$

The electric potential will be

$$\varphi(z_A) - \varphi(z_B) = -\int_{\bar{r}_B}^{\bar{r}_A} \bar{E} \cdot d\bar{\ell} = +\int_{z_A}^{z_B} dz\,\frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{2\varepsilon_0}(z_B - z_A) \qquad (z_B > z_A > 0)$$

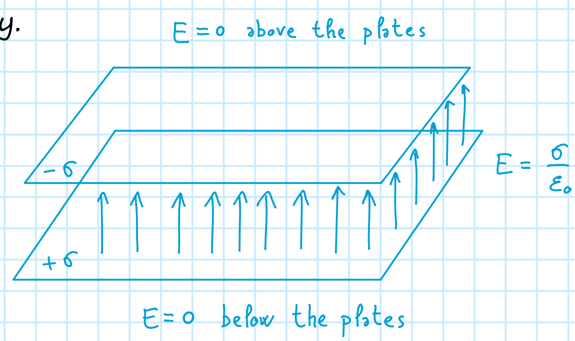
Therefore



$$E(z) = -\frac{\partial}{\partial z}\varphi = -\frac{\partial}{\partial z}\left(\frac{\sigma}{2\varepsilon_0}z\right) = -\frac{\sigma}{2\varepsilon_0}$$

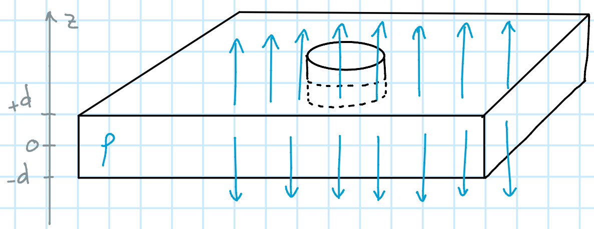
$$E(z) = -\frac{\partial}{\partial z}\varphi = -\frac{\partial}{\partial z}\left(-\frac{\sigma}{2\varepsilon_0}z\right) = \frac{\sigma}{2\varepsilon_0}$$

One can employ what obtained above to described the case of two parallel plates with opposite charge density.



Plane slab

One can obtain the electric field for a plane surface of uniform charge density starting from the case of a plane slab of non zero thickness and constant charge density per unit volume. Here we take the gaussian surface to be a cylinder with one of the flat faces in the slab (at $z = 0$) and one outside the slab.



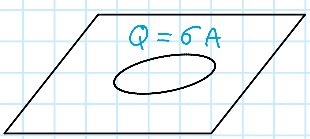
$$\bar{E} = E(z)\,\hat{k}$$

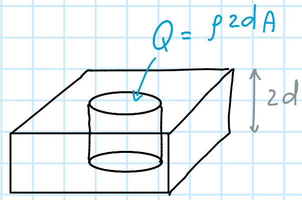
$$\varphi(z) = +\int_{\partial V} \bar{E}\cdot d\bar{s} = \begin{cases} \dfrac{\rho z A}{\varepsilon_0} & \text{if } z < d \\[2ex] \dfrac{\rho d A}{\varepsilon_0} & \text{if } z > d \end{cases}$$

$$+\,E(z)\,A + \underbrace{E(o)}_{\substack{=0 \\ \text{symmetry}}}\,A = \begin{cases} \dfrac{\rho z A}{\varepsilon_0} \\[2ex] \dfrac{\rho d A}{\varepsilon_0} \end{cases}$$

$$E(z) = \begin{cases} \dfrac{\rho z}{\varepsilon_0} & z < d \\[2ex] \dfrac{\rho d}{\varepsilon_0} & z > d \end{cases}$$

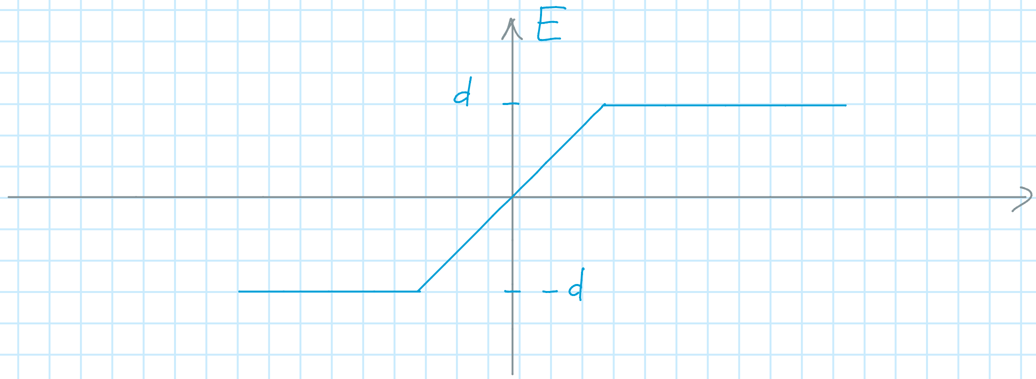
Now observe that one can write the charge density per unit surface as a function of ρ and d



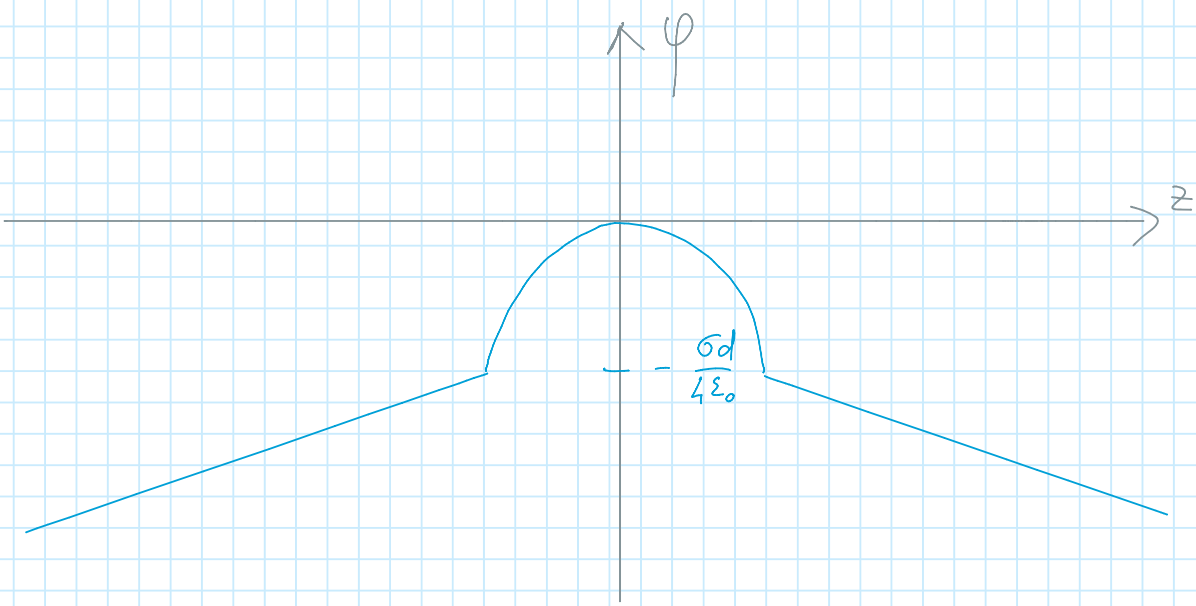


$$\longrightarrow \qquad \sigma = 2\rho d$$

$$E(z) = \begin{cases} \dfrac{\sigma z}{2 d \varepsilon_0} & z < d \\[2ex] \dfrac{\sigma}{2\varepsilon_0} & z > d \end{cases} \xrightarrow{d \to 0} \frac{\sigma}{2\varepsilon_0}$$



$$\varphi(z) = \begin{cases} -\dfrac{\sigma z^2}{4 d \varepsilon_0} & 0 < z < d \\[2ex] -\dfrac{\sigma}{2\varepsilon_0} z + \dfrac{\sigma d}{4 \varepsilon_0} & 0 < d < z \end{cases}$$

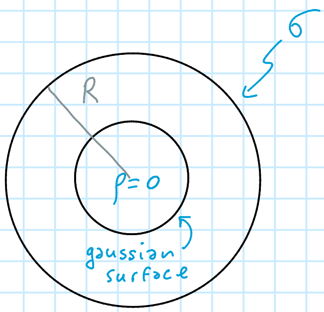


Empty spherical shell

Suppose that a charge is uniformly distributed on a spherical shell of radius R. The field is bound to be spherically symmetric inside and outside the shell. Outside the shell the field is the same as the field generated by a point like charge

$$Q = 4\pi R^2 \sigma$$

Inside the shell the field is simply 0 since there is no charge in a spherical gaussian surface included in the spherical shell.



$$\int_S \bar{E} \cdot d\bar{S} = E(r)\, 4\pi r^2 = \begin{cases} \dfrac{Q}{\varepsilon_0} & r > R \\ 0 & r < R \end{cases}$$

Therefore the potential inside the shell is constant.