Gradient

Calculate the gradient of the functions

$$f(x,y,z) = x^2 + 3xz^2 - 2xz\,,$$
$$g(x,y,z) = y^2 + 2xy + z^2 + 2yz\,.$$

Divergence and Curl

Calculate the divergence and the curl of the vectors

$$\mathbf{v}_a = (3z^4 - 2y)\hat{\mathbf{i}} - 3y\hat{\mathbf{j}} + yz^2\hat{\mathbf{k}}\,,$$
$$\mathbf{v}_b = x\hat{\mathbf{i}} + (z - y^2)\hat{\mathbf{j}} + xz\hat{\mathbf{k}}\,,$$
$$\mathbf{v}_c = yz\hat{\mathbf{i}} + zy\hat{\mathbf{j}} + xy\hat{\mathbf{k}}\,.$$

Divergence Theorem

Calculate the divergence of the vector

$$\mathbf{v} = (2xy + z^2)\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + 2xz\hat{\mathbf{k}}\,.$$

Calculate directly the volume integral

$$I_1 = \int_V d^3x\, \nabla \cdot \mathbf{v}\,,$$

over the cube V defined by the conditions

$$0 \le x \le 1 \qquad \text{and} \qquad 0 \le y \le 1 \qquad \text{and} \quad 0 \le z \le 1\,.$$

Calculate the surface integral

$$I_2 = \int_{\partial V} d\mathbf{S} \cdot \mathbf{v}\,,$$

over the surface of the cube V. Check that $I_1 = I_2$.

Curl Theorem

Calculate the curl of the vector

$$\mathbf{v} = 2yz\hat{\mathbf{i}} + 4xy^2\hat{\mathbf{j}}\,.$$

Calculate directly the surface integral

$$I_1 = \int_S d\mathbf{S} \cdot (\nabla \times \mathbf{v})\,,$$

where S is the surface of the square with corners placed at the points

$$\begin{aligned} P_1 &= (0,0,0)\,, \\ P_2 &= (1,0,0)\,, \\ P_3 &= (1,1,0)\,, \\ P_3 &= (0,1,0)\,. \end{aligned}$$

Calculate the line integral

$$I_2 = \int_C d\mathbf{l} \cdot \mathbf{v}\,,$$

C is the closed path along the edge of the square S. Chose the direction of $d\mathbf{S}$ and of the integration path C in a way which is consistent with Stokes' theorem. Check that $I_1 = I_2$.