Exact solution for the simple pendulum

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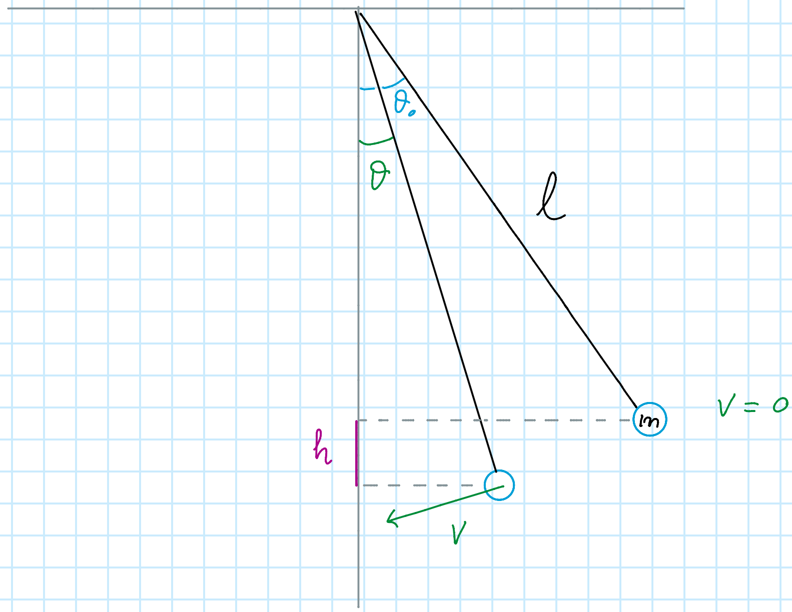
here we want to study the exact solution for the simple pendulum problem, without using the small angle approximation. As the pendulum drops, the potential energy lost is transformed in kinetic energy.



$$\frac{1}{2} m v^2 = m g h$$

$$v = \sqrt{2 g h} \qquad h = \ell \cos\vartheta - \ell \cos\vartheta_0$$

$$\ell \dot{\vartheta} = \sqrt{2 g \ell (\cos\vartheta - \cos\vartheta_0)}$$

Therefore

$$\frac{d\vartheta}{dt} = \sqrt{2 \frac{g}{\ell} (\cos\vartheta - \cos\vartheta_0)}$$

Now define

$$\omega_0^2 = \frac{g}{\ell}$$

and remember that

$$\cos\vartheta = \cos\left(\frac{\vartheta}{2}+\frac{\vartheta}{2}\right) = \cos^2\frac{\vartheta}{2} - \sin^2\frac{\vartheta}{2} = 1-2\sin^2\frac{\vartheta}{2}$$

$$\cos\vartheta - \cos\vartheta_0 = 1-2\sin^2\frac{\vartheta}{2} - 1 + 2\sin^2\frac{\vartheta_0}{2}$$

$$= 2\left(\sin^2\frac{\vartheta_0}{2} - \sin^2\frac{\vartheta}{2}\right)$$

So that

$$\boxed{\frac{d\vartheta}{dt} = \sqrt{4\omega_0^2}\,\sqrt{\sin^2\frac{\vartheta_0}{2} - \sin^2\frac{\vartheta}{2}}}$$

Now introduce the variable $x \equiv \omega_0 t$

$$\frac{d\vartheta}{dx} = 2\sqrt{\sin^2\frac{\vartheta_0}{2} - \sin^2\frac{\vartheta}{2}}$$

$$\left(\frac{d\vartheta}{dx}\right)^2 = 4\left(\sin^2\frac{\vartheta_0}{2} - \sin^2\frac{\vartheta}{2}\right)$$

Now introduce the quantities

$$k \equiv \sin\left(\frac{\vartheta_0}{2}\right)$$ MODULUS

$$y \equiv \frac{1}{k}\sin\left(\frac{\vartheta}{2}\right)$$

Then

$$\frac{dy}{dx} = \frac{1}{2k}\cos\left(\frac{\vartheta}{2}\right)\frac{d\vartheta}{dx}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4k^2}\cos^2\left(\frac{\vartheta}{2}\right)\left(\frac{d\vartheta}{dx}\right)^2$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{\cancel{4}\cancel{k}^2} \left(1 - k^2 y^2\right) \cancel{4} \left(\cancel{k}^2 - \cancel{k}^2 y^2\right)$$

$$= \left(1 - k^2 y^2\right)\left(1 - y^2\right)$$

$$= k^2 y^4 - \left(1 + k^2\right) y^2 + 1$$

The solution of the equation above with the initial condition $y(0) = 0$ is precisely the Jacobi elliptic function sn

$$y = \mathrm{sn}(x, k)$$

$$\sin\left(\frac{\theta}{2}\right) = k\, \mathrm{sn}(x, k)$$

$$\boxed{\theta(t) = 2 \arcsin\left[\sin\left(\frac{\theta_0}{2}\right) \mathrm{sn}\left(\omega_0 t, \sin\left(\frac{\theta_0}{2}\right)\right)\right]}$$

One can now plot this with Mathematica for arbitrary θ_0 as a function of t.

For small initial angles the function above is very close to

$$\theta(t) = \theta_0 \sin(\omega_0 t)$$

Now it is interesting to invert the function, the inverse of the elliptic sn function is the incomplete elliptic integral of the first kind F

$$\boxed{F(x, k) \equiv \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}}$$

INCOMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND

$$x = \mathrm{sn}(u, k)$$

$$F(x, k) = u \longrightarrow F(\mathrm{sn}(u, k), k) = u$$

RELATION BETWEEN sn AND F

Therefore

$$\sin\frac{\vartheta}{2} = \sin\left(\frac{\vartheta_0}{2}\right)\ \mathrm{sn}\left(\omega_0 t, \sin\frac{\vartheta_0}{2}\right)$$

$$\frac{\sin\frac{\vartheta}{2}}{\sin\frac{\vartheta_0}{2}} = \mathrm{sn}\left(\omega_0 t, \sin\frac{\vartheta_0}{2}\right)$$

$$F\left(\frac{\sin\frac{\vartheta}{2}}{\sin\frac{\vartheta_0}{2}}, \sin\frac{\vartheta_0}{2}\right) = \underbrace{F\left(\mathrm{sn}\left(\omega_0 t, \sin\frac{\vartheta_0}{2}\right)\right)}_{\omega_0 t}$$

$$\boxed{t = \frac{1}{\omega_0} F\left(\frac{\sin\frac{\vartheta}{2}}{\sin\frac{\vartheta_0}{2}}, \sin\frac{\vartheta_0}{2}\right)}$$

Remember that the initial condition was chosen so that the oscillation starts from $\theta = 0$ at the time $t = 0$.

The period is therefore 4 time the time that it takes to the pendulum bob to go from $\theta = 0$ to $\theta = \theta 0$.

$$T = \frac{4}{\omega_0} \underbrace{F\left(1, \sin\frac{\vartheta_0}{2}\right)}_{K\left(\sin\frac{\vartheta_0}{2}\right)}$$

COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

$$K(m) \equiv \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

$$\boxed{T = 4\sqrt{\frac{\ell}{g}}\; K\left(\sin\frac{\theta_0}{2}\right)}$$