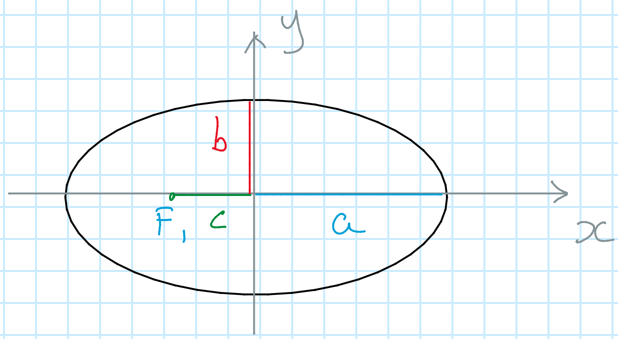
Elliptic functions as Trigonometry

Wednesday, August 26, 2020 7:33 AM

Notes take from the lectures by professor William Schwalm

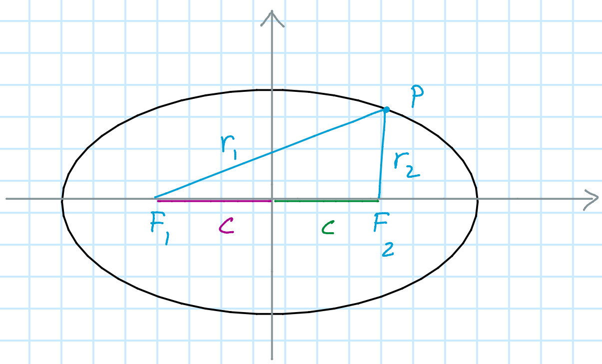
Start by considering an ellipse, there are two ways to look at it.



$$\boxed{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1}$$

ELLIPSE IN CARTESIAN COORDINATES

by definition c = distance FOCAL POINT to CENTER



$$\boxed{r_1 + r_2 = \text{const}}$$

ELLIPSE AS A A FUNCTION OF THE DISTANCE FROM THE FOCUSES

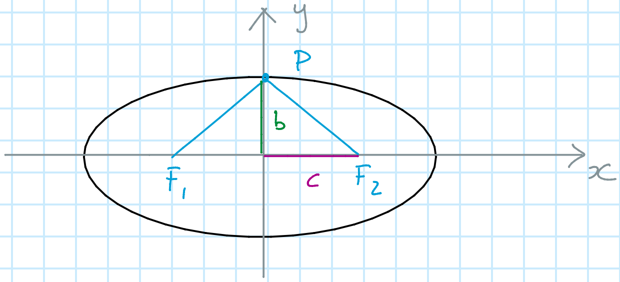
If one puts P on the x axis

$$r_1 = a + c$$

$$r_2 = a - c$$

$$r_1 + r_2 = 2a$$

If one puts P on the y axis



$$\overline{PF_2}^2 = \overline{PF_1}^2 = c^2 + b^2$$

but also

$$\overline{PF}_1 + \overline{PF}_2 = r_1 + r_2 = 2a = 2\sqrt{b^2+c^2}$$

Therefore

$$\boxed{a^2 = b^2 + c^2}$$

Finally define

$$\boxed{\epsilon = \frac{c}{a}}$$ ECCENTRICITY

And set $b \equiv 1$

So that

$$\boxed{c = \sqrt{a^2 - 1}}$$

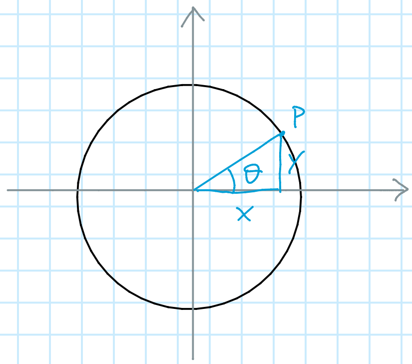
Modulus

One can define a parameter that regulates the shape of the ellipse

$$\boxed{k = \frac{\sqrt{a^2-1}}{a}}$$ MODULUS

Jacobi elliptic functions

For trig functions



$$x^2 + y^2 = 1$$

$$\sin\vartheta = y$$

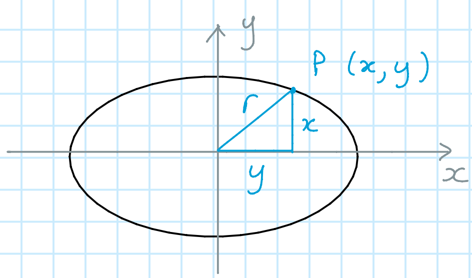
$$\cos\vartheta = x$$

$$\cos^2\vartheta + \sin^2\vartheta = 1$$

For ellipses (with $b = 1$)

$$\left(\frac{x}{a}\right)^2 + y^2 = 1$$

$$x^2 + y^2 = r^2$$



JAKOBI ELLIPTIC FUNCTIONS

$$\boxed{\begin{aligned} sn(u,k) &\equiv y \\ cn(u,k) &\equiv \frac{x}{a} \end{aligned}}$$

So that

$$cn^2(u,k) + sn^2(u,k) = 1$$

If one deals with a single ellipse, so that k is fixed, one can simplify the notation

$$sn(u,k) \to sn(u) \to sn$$

$$cn(u,k) \to cn(u) \to cn$$

For ellipses one define an extra trigonometric function

$$\boxed{dn\,(u,k) \equiv \frac{r}{a}}$$

3rd JACOBI ELLIPTIC FUNCTION

since $1 \le r \le a$

$$\frac{1}{a} \le dn \le 1$$

What is u ?

at $\vartheta = 0$ $(u = 0)$ $dn = 1$

at $\vartheta = \frac{\pi}{2}$ $dn = \frac{1}{a}$

From the definition of modulus

$$k^2 = \frac{a^2 - 1}{a^2} \qquad a^2 k^2 = a^2 - 1$$

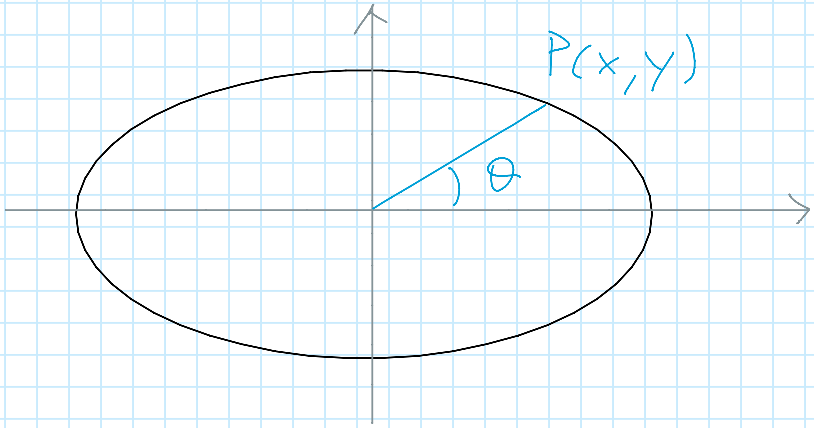
$$a^2 (k^2 - 1) = -1 \qquad a^2 = \frac{1}{1 - k^2} \qquad \boxed{a = \frac{1}{\sqrt{1 - k^2}}}$$

Definition of u

Summary so far

$$\boxed{\begin{aligned} sn\,(u,k) &= y \\ cn\,(u,k) &= \frac{x}{a} \\ dn\,(u,k) &= \frac{r}{a} \end{aligned}}$$

JACOBI'S ELLIPTIC FUNCTIONS



Define u in such a way that

$$\boxed{du = r\,d\vartheta}$$

Geometrical identities

$$\left(\frac{x}{a}\right)^2 + y^2 = 1 \longrightarrow \boxed{cn^2 + sn^2 = 1}$$

$$x^2 + y^2 = r^2 \longrightarrow a^2 cn^2 + sn^2 = a^2 dn^2$$

$$a^2(1 - sn^2) + sn^2 = a^2 dn^2$$

$$1 + \frac{1-a^2}{a^2} sn^2 = dn^2$$

$$1 - k^2 sn^2 = dn^2$$

$$\boxed{dn^2 + k^2 sn^2 = 1}$$

Or by replacing sn with cn

$$dn^2 + k^2(1 - cn^2) = 1$$

$$\boxed{dn^2 - k^2 cn^2 = 1 - k^2}$$

Derivatives of the elliptic functions

$$\tan\theta = \frac{y}{x} \qquad\qquad y = r\sin\theta$$
$$x = r\cos\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \qquad\qquad \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{1}{\sin\theta}\frac{dy}{d\theta} = \frac{dr}{d\theta} + r\frac{\cos\theta}{\sin\theta} = \frac{dr}{d\theta} + \frac{x}{\sin\theta}$$

$$\frac{1}{\cos\theta}\frac{dx}{d\theta} = \frac{dr}{d\theta} - \frac{y}{\cos\theta}$$

$$\frac{1}{\sin\theta}\frac{dy}{d\theta} - \frac{1}{\cos\theta}\frac{dx}{d\theta} = \frac{x}{\sin\theta} + \frac{y}{\cos\theta}$$

Multiply by $r\cos\theta\sin\theta$

$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = x^2 + y^2$$

$$x\,dy - y\,dx = (x^2 + y^2)\,d\theta$$

$$\boxed{d\theta = \frac{x\,dy - y\,dx}{r^2}}$$

$$r\,d\theta = \frac{x\,dy}{r} - \frac{y\,dx}{r} = du$$

Now from

$$\frac{x^2}{a^2} + y^2 = 1 \qquad \frac{x\,dx}{a^2} + y\,dy = 0$$

$$dx = -a^2 \frac{y}{x}\,dy$$

And plugging dx back into the previous relation

$$du = \frac{x\,dy}{r} - \frac{y}{r}\left(-a^2 \frac{y}{x}\,dy\right)$$

$$du = \frac{x^2\,dy}{rx} + a^2 \frac{y^2}{rx}\,dy$$

$$du = \frac{x^2 + a^2y^2}{rx}\,dy = \frac{a}{x}\,\frac{a}{r}\,dy$$

$$x^2 + a^2y^2 = a^2$$

Finally

$$\frac{dy}{du} \equiv \frac{d}{du}\,sn = \frac{xr}{a^2} = cn\ dn$$

$$\boxed{\frac{d}{du}\, sn = cn\ dn}$$

We can find another differential relation by differentiating

$$cn^2 + sn^2 = 1$$

$$2\,cn\,\frac{d\,cn}{du} + 2\,sn\,\frac{d\,sn}{du} = 0$$

$$cn\,\frac{d\,cn}{du} + sn\left(cn\ dn\right) = 0$$

$$\boxed{\frac{d\,cn}{du} = -\,sn\ dn}$$

Then one can differentiate the relation

$$dn^2 - k^2 cn^2 = 1 - k^2$$

$$2\,dn\,\frac{d}{du}dn - 2k^2 cn\,\frac{d}{du}cn = 0$$

$$dn\,\frac{d}{du}dn - k^2 cn\left(-\,sn\ dn\right) = 0$$

$$\boxed{\frac{d}{du}dn = -\,k^2 cn\ sn}$$

By differentiating one of the Jacobi elliptic functions one gets a product of the other two.

Tangent type functions

In modern notation

$$\boxed{\frac{sn(u,k)}{cn(u,k)} \equiv sc(u,k)}$$

"TANGENT"

$$\boxed{\frac{1}{sn(u,k)} \equiv ns(u,k)}$$

"SECANT"

$$cd(u,k) \equiv \frac{cn(u,k)}{dn(u,k)} = \frac{1}{dc(u,k)}$$

The ratio of two functions is indicated by the first letter of the numerator together with the first letter of the denominator.
The inverse of a function is indicated by reversing the letters.

Elliptic functions can be used to solve non linear differential equations

There are 12 elliptic functions

$$\begin{array}{lll} sn, & cn, & dn \\ ns, & nc, & nd \\ sc, & sd, & cd \\ cs, & ds, & dc \end{array}$$

All of these functions satisfy a specific non linear differential equation.
For example

$$\frac{d}{du} sn = cn\, dn$$

$$\left(\frac{d}{du} sn\right)^2 = cn^2\, dn^2$$

$$= (1 - sn^2)(1 - k^2 sn^2)$$

$$\boxed{\left(\frac{d}{du}\, sn\right)^2 = k^2 sn^4 - (k^2+1)\, sn^2 + 1}$$

Non linear polynomial differential equation.
The general form of these equations for any of these functions are

$$\left(\frac{d}{du}\, zn\right)^2 = \alpha\, zn^4 + \beta\, zn^2 + \gamma$$

zn is a generic Jacobi elliptic function

α, β, γ are functions of k

For sn

$$\alpha = k^2 \qquad \beta = -(1+k^2) \qquad \gamma = 1$$

For example the equation for cn will be

$$\left(\frac{d\, cn}{du}\right)^2 = sn^2\, dn^2$$

$$dn^2 = 1 - k^2 + k^2 cn^2 \qquad sn^2 = 1 - cn^2$$

$$\left(\frac{d\, cn}{du}\right)^2 = (1 - cn^2)(1 - k^2 + k^2 cn^2)$$

$$= -k^2 cn^4 + (k^2 - 1 + k^2)\, cn^2 + (1 - k^2)$$

$$= -k^2 cn^4 - (1 - 2k^2)\, cn^2 + 1 - k^2$$

Second order differential equations

The first order differential equations can be rewritten as second order equations

$$\frac{d}{du}\left(\frac{dz_n}{du}\right)^2 = 2\,\frac{dz_n}{du}\,\frac{d^2 z_n}{du^2}$$

$$= \left(4\alpha z_n^3 + 2\beta z_n\right)\frac{dz_n}{du}$$

Therefore

$$\boxed{\frac{d^2 z_n}{du^2} = 2\alpha z_n^3 + \beta z_n}$$