Constrained systems: general case

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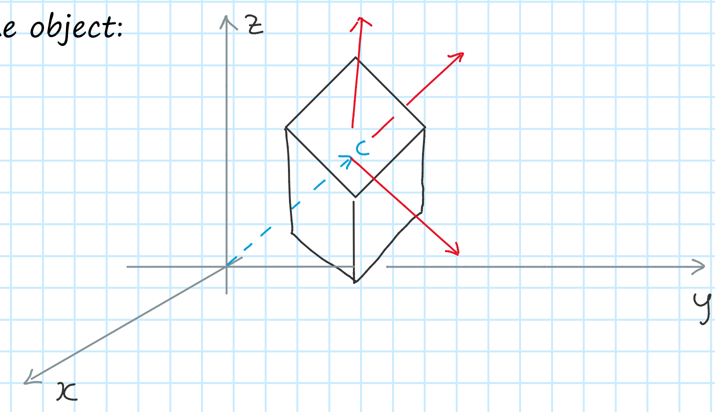
In a system with N particles the positions of all of the particles can in general be described by specifying n generalized coordinates. Notice that n can be different from (and smaller than) 3N

$$\bar{r}_i = \bar{r}_i\,(q_1, q_2, \dots q_n, t) \qquad i = 1, 2, \dots, N$$

The generalized coordinates can be written as a function of the positions, by inverting the relation above

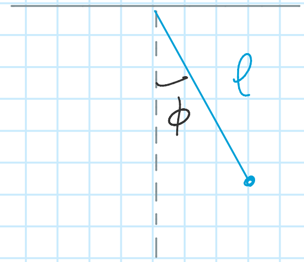
$$q_i = q_i\,(\bar{r}_1, \bar{r}_2, \dots, \bar{r}_N, t) \qquad i = 1, 2, \dots, n$$

One needs to choose generalized coordinates such that n is the smallest number of generalized coordinates for which the relations above can be written. For a system of N particles free to move in a 3-dimensional space one needs 3N generalized coordinates. However, for many constrained systems the number of generalized coordinates is much smaller than 3N. A striking example of this situation is the case of a rigid body. The number of particle in a macroscopic rigid body is astronomically big (remember that Avogadro's number is ~ 10^23) but, since the reciprocal positions of the molecules are fixed, one needs only 6 generalized coordinates to describe the position of the object. Three generalized coordinates describe the position of the center of mass of the system, three angles describe the orientation of the object:



Simple pendulum

In order to illustrate the nature of the relations between positions and generalized coordinates we consider a few simple examples. The simplest of then all is probably the simple (!) pendulum, that we already considered. We know that the position of a simple pendulum can be described by a single generalized coordinate, which can be chosen to be the angle with respect to the vertical direction



$$\bar{r} = (x, y) = (\ell \sin\phi, \ell \cos\phi)$$

$$\underbrace{N}_{\text{\# of } \bar{r} \text{ vectors}} = 1 \qquad \underbrace{2N}_{\text{on a plane, 2 components for each } \bar{r}} = 2 \qquad \underbrace{q = \phi}_{n = 1}$$

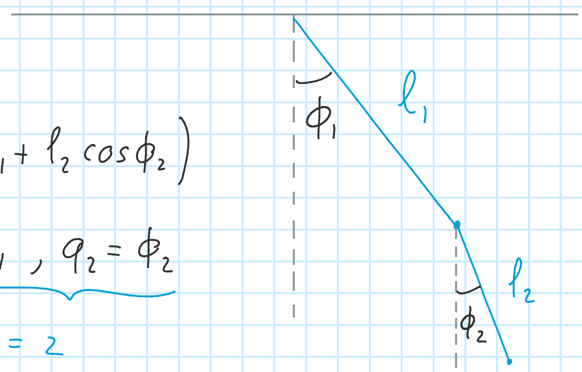
Double pendulum

If one attaches a second pendulum to the pendulum bob of the first one, the system is characterized by two generalized coordinates

$$\bar{r}_1 = (\ell_1 \sin\phi_1, \ell_1 \cos\phi_1)$$

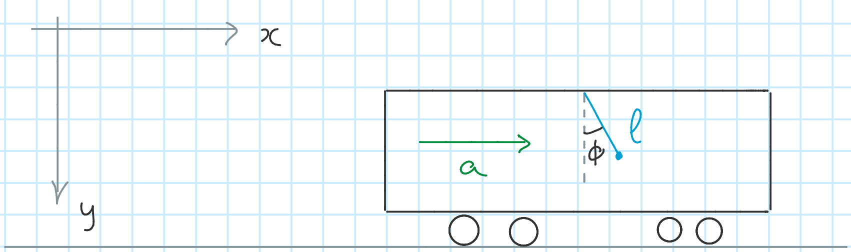
$$\bar{r}_2 = (\ell_1 \sin\phi_1 + \ell_2 \sin\phi_2, \ell_1 \cos\phi_1 + \ell_2 \cos\phi_2)$$

$$N = 2, \quad 2N = 4 \qquad \underbrace{q_1 = \phi_1, \; q_2 = \phi_2}_{n = 2}$$



Pendulum on an accelerating railway car

In the two examples considered above the relations between positions and generalized coordinates do not depend on time. There are cases in which these relations do depend on time. Consider a pendulum hanging from the ceiling of a railroad car that is accelerating along the tracks. We can't describe the system by using a frame of reference at rest with respect to the car, because that would not be an inertial frame of reference (the car is accelerating). Let's then choose a frame of reference at rest with respect to the tracks.



$$\bar{r} = (x, y) = \left(\ell \sin\phi + \tfrac{1}{2} a t^2 + v_0 t + x_0,\ \ell \cos\phi \right)$$

$$N = 1 \qquad 2N = 2 \qquad \underbrace{q = \phi}_{n = 1}$$

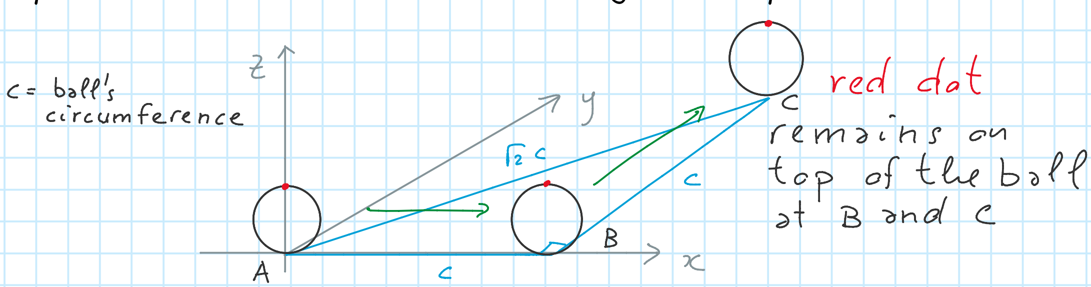
Generalized coordinates for which the relations between the positions and the generalized coordinates do not involve time are called natural.

$\boxed{\text{Degrees of freedom}}$

The degrees of freedom of a system are defined as the number of generalized coordinates that can be independently varied in a small displacement of the system.

An unconstrained system of N particles is characterized by 3N generalized coordinates. If a system of N particles is described by less than 3N generalized coordinates, the system is said to be constrained.

Systems for which the number of generalized coordinates is equal to the number of degrees of freedom are called holonomic systems. All of the systems that we will deal with are holonomic systems. However it is important to realize that there are non holonomic systems (even simple non holonomic systems). A non holonomic system is a system whose state depends on the path taken to achieve it. A classic example is a sphere free to roll (without slipping) on a flat surface. From any given position the ball can only move in two independent directions, x and y. Consequently, according to the definition adopted, the ball has two degrees of freedom. However, one can prove that to describe the configuration of the ball one needs more than two generalized coordinates. In fact, one can mark the point on top of the ball and then roll the ball along a closed path as follows:



However, after the last leg of the trip, from C back to A, the red dot will not be on top of the ball any longer. So in spite of the fact that at the end of the trip the ball have the same position in the x y plane as at the beginning of its journey, the configuration of the ball is not the same. The ball has a different orientation at the end of the trip with respect to the beginning. This system has two degrees of freedom, but one needs five generalized coordinates to describe its configuration. Consequently, this system is non holonomic.

Lagrange equations

It is possible to prove that holonomic systems satisfy a Lagrange's equations for every generalized coordinate q_i

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$