Kronecker and Levi Civita symbols

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Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for $i, j \in \{1, 2, 3\}$

$$\delta_{11} = \delta_{22} = \delta_{33} = 1$$

$$\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0$$

Levi-Civita epsilon

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123,\ 312,\ 231 \\ -1 & \text{if } ijk = 321,\ 132,\ 213 \\ 0 & \text{if two indices are equal} \end{cases}$$

Notice that

$$\varepsilon_{ijk} = -\varepsilon_{jik}$$

The Levi-Civita tensor is antisymmetric in the exchange of the indices

The Levi-Civita symbol if very useful to write the components of a cross product

$$(\bar{a} \times \bar{b})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} a_j b_k = \varepsilon_{ijk} a_j b_k \leftarrow$$

repeated indices are summed

Sums over indices

$$\varepsilon_{ijk} \varepsilon_{ilm} = \varepsilon_{1jk} \varepsilon_{1lm} + \varepsilon_{2jk} \varepsilon_{2lm} + \varepsilon_{3jk} \varepsilon_{3lm}$$

Only one of the three terms above can be different from zero, let's assume it is the first, then one has

$$j \neq k \neq 1 \qquad \ell \neq m \neq 1$$

$\boxed{\text{if } j=2} \rightarrow k=3 \qquad \varepsilon_{ijk}\,\varepsilon_{i\ell m} = \varepsilon_{123}\,\varepsilon_{1\ell m}$

$$= \begin{cases} \varepsilon_{123}\,\varepsilon_{123} = 1 & \text{if } \ell=2 \\ \varepsilon_{123}\,\varepsilon_{132} = -1 & \text{if } \ell=3 \end{cases}$$

therefore $j=2,\ k=3,\ \ell=2,\ m=3$

$$\varepsilon_{ijk}\,\varepsilon_{i\ell m} = 1$$

$j=2,\ k=3,\ \ell=3,\ m=2 \qquad \varepsilon_{ijk}\,\varepsilon_{i\ell m} = -1$

These results (and the results for all of the other possible choices of the indices) can be written in a simple way

$$\boxed{\varepsilon_{ijk}\,\varepsilon_{i\ell m} = \delta_{j\ell}\,\delta_{km} - \delta_{jm}\,\delta_{\ell k}}$$

if $j=2,\ k=3,\ \ell=2,\ m=3 \qquad \delta_{22}\,\delta_{33} - \delta_{23}\,\delta_{23} = 1$

if $j=2,\ k=3,\ \ell=3,\ m=2 \qquad \delta_{23}\,\delta_{23} - \delta_{22}\,\delta_{33} = -1$