Properties of the center of mass

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Rigid body:

A rigid body is a collection of N particles with the property that the shape cannot change, i.e. The distance between any two of its constituents particles is fixed.

Of course real physical object are not perfectly rigid bodies. However, many physical objects are sufficiently close to an ideal rigid body so that the rigid body model is perfectly suited to describe the motion of these physical objects.

One should notice that a rigid body made of N particles is a system whose configuration is described by only 6 numbers: The coordinates of its center of mass and three angles that will describe the orientation of the body. In contrast, the configuration of a generic non rigid collection of N particles is described by the coordinates of each one of the particles, for a total of 3N coordinates.

Let's review the definition of center of mass, calculated with respect of the origin of the reference frame O

$$\bar{R} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \bar{r}_\alpha$$

CENTER OF MASS OF A DISCRETE MASS DISTRIBUTION

$$\bar{R} = \frac{1}{M} \int_V \bar{r}\, dm = \frac{1}{M} \int_V \bar{r} \rho(\bar{r})\, dV$$

C.O.M OF A CONTINUOUS MASS DISTRIBUTION

Total c.o.m. Momentum

The c.o.m momentum is the sum of the momenta of the individual particles

$$\bar{P} = \sum_{\alpha=1}^{N} \bar{p}_\alpha = \sum_\alpha m_\alpha \dot{\bar{r}}_\alpha = M \dot{\bar{R}}$$

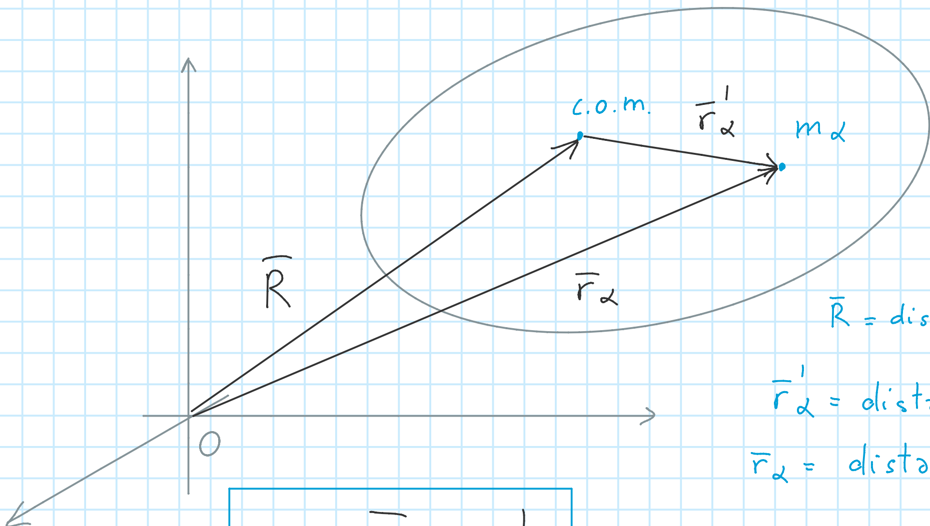
The total momentum of the system is therefore equal to the momentum of a single particle of mass M moving at the velocity of the center of mass.

By differentiating the total momentum with respect to time, it is possible to see that Newton's equation for the center of mass is

$$M\ddot{\bar{R}} = \sum_i \bar{F}_i = \bar{F}_{ext}$$

The center of mass accelerates as if the total mass of the system was concentrated in the center of mass and as if this mass was subject to a net force which is the total external force acting on the system. Indeed notice that, because of Newton's third law, the internal forces among the particles that make up the rigid body cancel out in the sum above.

Total angular momentum



$\bar{R}$ = distance origin - c.o.m.

$\bar{r}_\alpha'$ = distance m_α - c.o.m

$\bar{r}_\alpha$ = distance m_α - origin

$$\boxed{\bar{r}_\alpha = \bar{R} + \bar{r}_\alpha'}$$

The angular momentum of the particle α with respect to the origin of the frame of reference is then

$$\bar{\ell}_\alpha = \bar{r}_\alpha \times \bar{p}_\alpha = \bar{r}_\alpha \times m_\alpha \dot{\bar{r}}_\alpha$$

The total angular momentum of the rigid body with respect to O is then

$$\bar{L} = \sum_{\alpha=1}^{N} \bar{\ell}_\alpha = \sum_{\alpha=1}^{N} \bar{r}_\alpha \times m_\alpha \dot{\bar{r}}_\alpha$$

$$= \sum_{\alpha=1}^{N} \bar{R} \times m_\alpha \dot{\bar{R}} + \sum_\alpha \bar{R} \times m_\alpha \dot{\bar{r}}_\alpha' + \sum_\alpha \bar{r}_\alpha' \times m_\alpha \dot{\bar{R}} + \sum_\alpha \bar{r}_\alpha' \times m_\alpha \dot{\bar{r}}_\alpha'$$

$$= \bar{R} \times M\dot{\bar{R}} + \bar{R} \times \left(\sum_\alpha m_\alpha \dot{\bar{r}}_\alpha'\right) + \left(\sum_\alpha m_\alpha \bar{r}_\alpha'\right) \times \dot{\bar{R}} + \sum_\alpha \bar{r}_\alpha' \times m_\alpha \dot{\bar{r}}_\alpha'$$

Now one can observe that the second and third terms in the sum above is zero

$$\sum_{\alpha=1}^{N} m_\alpha \bar{r}_\alpha' = M\bar{R}'$$

$\bar{R}'$ = location of the c.o.m. calculated w.r.t a frame located in the c.o.m.

$$\Longrightarrow \quad \bar{R}' = 0$$

$$\sum_{\alpha=1}^{N} m_\alpha \dot{\bar{r}}_\alpha' = M\dot{\bar{R}}'$$

$\dot{\bar{R}}'$ = velocity of the c.o.m. w.r.t the c.o.m.

$$\Longrightarrow \quad \dot{\bar{R}}' = 0$$

Therefore

$$\bar{L} = \bar{R} \times M\dot{\bar{R}} + \sum_{\alpha=1}^{N} \bar{r}_\alpha' \times m_\alpha \dot{\bar{r}}_\alpha'$$

$$= \underbrace{\bar{R} \times \bar{P}}_{\text{angular momentum of the c.o.m. w.r.t } O} + \underbrace{\sum_{\alpha=1}^{N} \bar{r}_\alpha' \times m_\alpha \dot{\bar{r}}_\alpha'}_{\text{angular momentum of the rigid body w.r.t the c.o.m.}}$$

If one considers for example a planet orbiting the sun, so that the c.o.m. of the system corresponds approximately with the sun, the relation above tells us that the total angular momentum of the planet is the vector sum of the orbital angular momentum and the spin angular momentum due to the revolution of the planet around its axis

$$\boxed{\bar{L} = \bar{L}_{orb} + \bar{L}_{spin}}$$

origin of the frame centered on the sun. C.o.m of the planet at $\bar{R}$

In the system sun-planet, the orbital and spin angular momentum are separately conserved. Indeed

$$\dot{\bar{L}}_{orb} = \underbrace{\dot{\bar{R}} \times \bar{P}}_{\dot{\bar{R}} \times M\dot{\bar{R}} = 0} + \bar{R} \times \dot{\bar{P}} = \underbrace{\bar{R} \times \bar{F}_{ext}}$$

Torque applied by the external forces on the planet c.o.m., calculated w.r.t the position of the sun

If the external force is a central force, then the torque is zero and the orbital angular momentum is conserved

$$\dot{\bar{L}}_{orb} = 0$$

Gravity is a central force so the orbital angular momentum of the planet is conserved (if we neglect small effects due for example to the fact that the planets and the sun are not perfect spheres)

One can deal with the spin angular momentum as follows

$$\dot{\bar{L}}_{spin} = \dot{\bar{L}} - \dot{\bar{L}}_{orb}$$

$$\dot{\bar{L}} = \underbrace{\sum_{\alpha=1}^{N} \bar{r}_\alpha \times \bar{F}_\alpha^{ext}}_{\text{internal forces do not contribute to the total torque}} = \sum_{\alpha=1}^{N} (\bar{r}_\alpha' + \bar{R}) \times \bar{F}_\alpha^{ext}$$

$$= \sum_{\alpha=1}^{N} \bar{r}_\alpha' \times \bar{F}_\alpha^{ext} + \underbrace{\bar{R} \times \bar{F}^{ext}}_{\dot{\bar{L}}_{orb}}$$

$$\dot{\bar{L}}_{spin} = \dot{\bar{L}} - \dot{\bar{L}}_{orb} = \sum_{\alpha=1}^{N} \bar{r}_\alpha' \times \bar{F}_\alpha^{ext} = \bar{\Gamma}^{ext}_{c.o.m.}$$

torque about the c.o.m. of the planet

Notice that the reference frame attached to the center of mass is in general not an inertial frame (ex. The center of mass of a planet is orbiting around the sun and therefore has a centripetal acceleration). However, the equation above shows that the rotational version of Newton's second law still works

$$\dot{\bar{L}}_{spin} = \bar{\Gamma}^{ext}_{c.o.m.}$$

In the case of a planet and the sun, the external torque about the center of mass is small and the spin angular momentum is almost constant.

$\boxed{\text{Kinetic energy}}$

The kinetic energy of a collection of N particles is

$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\bar{r}}_\alpha^2 = \sum_\alpha \frac{1}{2} m_\alpha \left(\dot{\bar{R}} + \dot{\bar{r}}'_\alpha\right)^2 = \sum_\alpha \frac{m_\alpha}{2}\left(\dot{\bar{R}}^2 + (\dot{\bar{r}}'_\alpha)^2 + 2\dot{\bar{R}}\cdot\dot{\bar{r}}'_\alpha\right)$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \dot{\bar{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha (\dot{\bar{r}}'_\alpha)^2 + \dot{\bar{R}}\cdot\underbrace{\sum_\alpha m_\alpha \dot{\bar{r}}'_\alpha}_{=0}$$

$$= \underbrace{\frac{1}{2} M \dot{\bar{R}}^2}_{\text{Kinetic energy of the motion of the center of mass}} + \underbrace{\frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha (\dot{\bar{r}}'_\alpha)^2}_{\text{Kinetic energy of the motion around the c.o.m.}}$$

The motion of a rigid body around the center of mass is rotational motion, so the last term in the sum is the rotational kinetic energy.

We can take a step back and see that the line below does not depend on the assumption that R is the location of the c.o.m.

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\bar{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha (\dot{\bar{r}}'_\alpha)^2 + \dot{\bar{R}}\cdot\sum_\alpha m_\alpha \dot{\bar{r}}'_\alpha$$

R could be any point. Now let's assume that R has zero velocity (even only zero velocity at a specific time). In that case the kinetic energy becomes

$$T = \frac{1}{2} \sum_{\alpha} m_\alpha \left(\dot{\bar{r}}'_\alpha\right)^2$$

ROTATIONAL ENERGY RELATIVE TO $\bar{R}$, ASSUMING $\dot{\bar{R}} = 0$

This shows for example that the kinetic energy of a bicycle wheel is the rotational kinetic energy of the wheel around the point of contact with the ground, since this point is instantaneously at rest.

Potential energy of a rigid body

Assume that the forces among the N particles of a rigid body are conservative. Then the total potential energy of the rigid body can be written as

$$U = U^{ext} + U^{int}$$

sum of the potential energies due to external forces

$$U^{int} = \sum_{\alpha<\beta} U_{\alpha\beta}(r_{\alpha\beta})$$

distance between particle α and particle β

Since in a rigid body the distances among particles is fixed, the internal potential energy is a constant and as such does not play a role in the equations of motion. Consequently it can be set equal to zero.