Falling mass phase space

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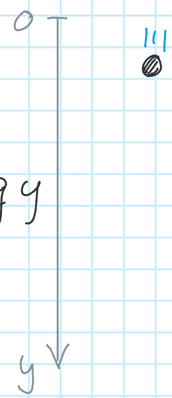
It is often interesting to follow the trajectories starting from points close to each other in phase space. One could think that by changing slightly the initial conditions the motions would not change drastically, o that phase space trajectories that are close to each other at some given moment will remain close to each other. For some systems this is indeed the case, but for other systems, called chaotic systems, a small change in the initial conditions leads to completely different phase space orbits. This is one of the reasons that make the study of phase space orbits interesting.

One can start by studying a very simple system, such as a mass in free fall

$$U = -mgy$$



Consider 4 different initial conditions

$$A_0 \rightarrow y_0 = p_0 = 0$$

$$B_0 \rightarrow y_0 = Y \qquad p_0 = 0$$

$$C_0 \rightarrow y_0 = Y \qquad p_0 = P$$

$$D_0 \rightarrow y_0 = 0 \qquad p_0 = P$$

The Hamiltonian and Hamilton's equations are

$$\mathcal{H} = T + U = \frac{p^2}{2m} - mgy$$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial y} = mg$$

The solutions of these equations are

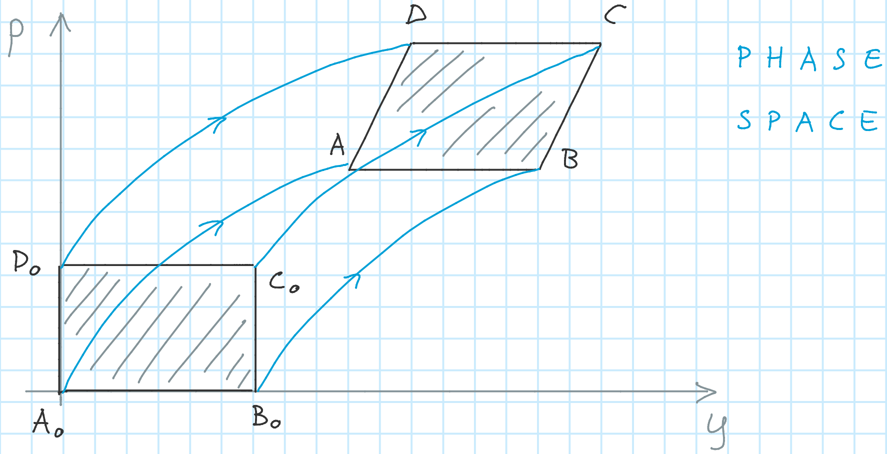
$$p = p_0 + mgt \qquad (\dot{y} = v_0 + gt)$$

$$y = y_0 + \frac{p_0}{m} t + \frac{1}{2} g t^2$$

These are obviously the free fall equations that we are familiar with from elementary physics. The trajectories in phase space are parabolas of the form

$$y = y_0 + \frac{p_0}{m} \frac{p - p_0}{g} + \frac{1}{2} g \left(\frac{p - p_0}{g} \right)^2$$

The phase space trajectories starting from the four initial conditions listed above look as follows



The areas of the rectangle and the parallelepiped are the same since they have the same base and the same height:

$$\overline{A_0 B_0} = Y$$

$$\overline{AB} = \underbrace{Y + \frac{1}{2} g t^2}_{\text{y coordinate of B at time t}} - \underbrace{\frac{1}{2} g t^2}_{\text{y coordinate of A}} = Y$$

$$\overline{A_0 D_0} = P$$

$$p_D - p_A = P + mgt - mgt = P$$

The fact that the areas of regions of the phase space do not change with time is am important general fact that is known as Liouville' s theorem.