Phase space

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It is not surprising that a system of n second order differential equations, the Lagrange equations, is equivalent to a system of 2 n first order differential equations, which are Hamilton equations. It can indeed be shown that any system of n second order differential equations can be rewritten as a system of 2n first order differential equations. For simplicity we show this in the case of a system that depends on a single generalized coordinate. A generic second order differential equation can be written as

$$f(\ddot{q}, \dot{q}, q) = 0$$

1 2nd order diff eq.

One can now introduce a second variable s such that

$$s \equiv \dot{q} \qquad\qquad f(\dot{s}, s, q) = 0$$

system of 2 1st order diff. eq.

In this sense Lagrangian and Hamiltonian formalisms lead to equations which are equivalent in complexity.

However, the Hamiltonian formalism can be interpreted as a system of first order differential equations that describes the motion of a point in phase space. Remember that the phase space is a space with 2 n dimensions where the coordinates are the generalized coordinates and the generalized momenta.

point in phase space

$$\longrightarrow \{\underbrace{q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n}_{2n \text{ coordinates}}\} \equiv \underbrace{\{\bar{q}, \bar{p}\}}_{\text{introduce vector notation to simplify the formalism}} \equiv \bar{z}$$

$$\left.\begin{aligned} \bar{q} &\equiv \{q_1, q_2, \ldots, q_n\} \\ \bar{p} &\equiv \{p_1, p_2, \ldots, p_n\} \end{aligned}\right\} \text{n dim vectors}$$

$$\bar{z} \equiv \{q_1, \ldots, q_n, p_1, \ldots, p_n\} \qquad \text{2n dim vector}$$

Hamilton equations can then be cast in the form

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = f_i(\bar{q}, \bar{p}) \qquad i = 1, \ldots, n \longrightarrow \dot{\bar{q}} = \bar{f}(\bar{q}, \bar{p})$$

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = g_i(\bar{q}, \bar{p}) \qquad i = 1, \ldots, n \longrightarrow \dot{\bar{p}} = \bar{g}(\bar{q}, \bar{p})$$

$$\longrightarrow \quad \bar{h} \equiv \{\bar{f}(\bar{q}, \bar{p}), \bar{g}(\bar{q}, \bar{p})\} = \bar{h}(\bar{z})$$

$$\boxed{\dot{\bar{z}} = \bar{h}(\bar{z})}$$

HAMILTON'S EQS AS FIRST ORDER DIFF. EQ. FOR A PHASE SPACE POINT

The symmetry between generalized coordinates and generalized momenta in the Hamiltonian formalism is an advantage of the Hamiltonian formalism over the Lagrangian formalism. In Lagrangian mechanics, Lagrange equations maintain the same form when one changes generalized coordinates. These changes in generalized coordinates correspond to a basis change in configuration space.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \xrightarrow{\bar{Q} = \bar{Q}(\bar{q})} \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Q}_i} - \frac{\partial \mathcal{L}}{\partial Q_i} = 0$$

Hamilton equations remain invariant in form for certain changes in the coordinates that span the phase space. These coordinates changes are called canonical transformations and mix generalized coordinates and generalized momenta

$$\bar{Q} = \bar{Q}(\bar{q}, \bar{p}) \qquad \bar{P} = \bar{P}(\bar{q}, \bar{p})$$

Canonical transformations are an important topic that we will not further discuss here.

Phase space orbits

At each instant in time the system occupies a point z in phase space. Hamilton equations determine the trajectory of the system in phase space. Since the phase space has 2 n dimensions it is difficult to visualize it (except in the case in which n = 1, or by looking at projections of it on a two dimensional plane, as it is done in Poincare maps).

It is important to notice that there is always only one trajectory going through a point in phase space. In other words, different phase space orbits cannot intersect each other. In fact each point in phase space can be taken as an initial condition for the system, and initial conditions and Hamilton equations determine unambiguously the future behavior of the system and, consequently the trajectory of the system in phase space.