Hamilton equations for a mass on a cone

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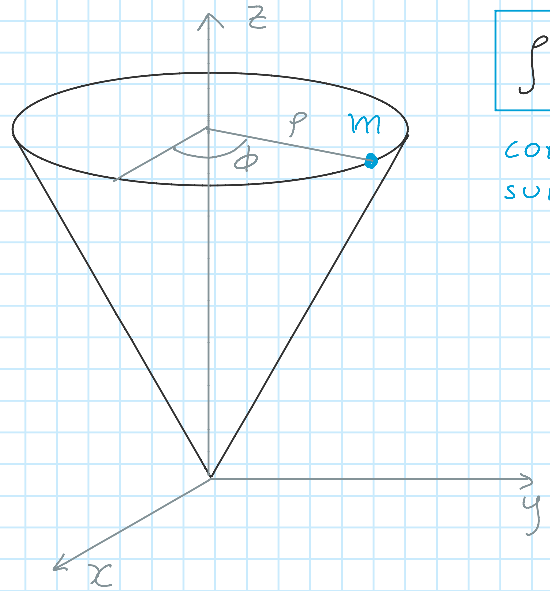
Consider a mass free to move on the surface of an upside down cone, fixed in space. The mass is subject only to gravity and to the constraint, i.e. it must move on the surface of the cone.



$$\boxed{\rho \equiv c\,z}$$

CONE SURFACE

One can analyze this situation with Hamiltonian tools, by using z and ϕ as generalized coordinates

$$T = \frac{1}{2} m \left[\dot{\rho}^2 + (\rho \dot{\phi})^2 + \dot{z}^2 \right] = \frac{1}{2} m \left[(c^2+1)\dot{z}^2 + (c z \dot{\phi})^2 \right]$$

$$U = m g z$$

The generalized momenta and the Hamiltonian are then

$$p_z = \frac{\partial T}{\partial \dot{z}} = m(c^2+1)\dot{z} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m c^2 z^2 \dot{\phi}$$

$$T = \frac{1}{2m}\left[\frac{p_z^2}{(c^2+1)} + \frac{p_\phi^2}{c^2 z^2} \right]$$

$$\boxed{\mathcal{H} = \frac{1}{2m}\left[\frac{p_z^2}{c^2+1} + \frac{p_\phi^2}{c^2 z^2} \right] + m g z}$$

Hamilton's equations are then

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m(c^2+1)} \qquad \dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = +\frac{p_\phi^2}{mc^2 z^3} - mg$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mc^2 z^2} \qquad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0$$

The last of the Hamilton equations above is simply the conservation of the z component of the angular momentum

$$p_\phi = m c^2 z^2 \dot{\phi} = m \rho^2 \dot{\phi} = \text{constant}$$

One can deduce a few of the motion properties by analyzing the Hamiltonian and Hamilton's equations of motion. Since the generalized coordinates are natural, the Hamiltonian is equal to the total energy, which is conserved. However, the last term in the Hamiltonian grows with z. Consequently there must be a maximum value for z. Similarly the second term in the Hamiltonian grows when z becomes small. Consequently, since the total energy is conserved, there must be a minimum value for z. For a given E

$$z_{min} \leq z \leq z_{max}$$

So unless the angular momentum is zero, the mass will not fall to the bottom of the cone. (Remember that we are not considering friction)
The equations of motion will also have a solution that corresponds to a circular trajectory, so that the mass will remain at a constant z. Indeed if one requires that the time derivative of z vanishes, one finds the angular momentum that corresponds to the circular orbit.

$$\dot{z} = 0 \longrightarrow p_z = 0 \longrightarrow \dot{p}_z = 0$$

$$\dot{p}_z = \frac{p_\phi^2}{mc^2 z^3} - mg = 0$$

$$p_\phi^2 = m^2 g c^2 z^3 \qquad p_\phi = \pm mc\sqrt{g z^3}$$