Bead on a rotating hoop

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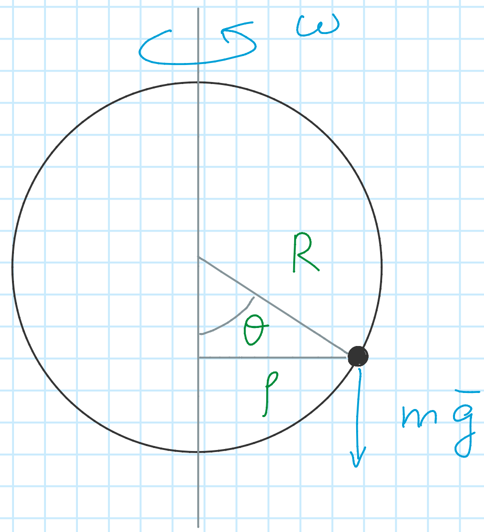
Let's consider the case of a bead mounted on a circular hoop. The hoop is vertical and it spins around a vertical axis going through its center. The angular velocity of the rotation of the hoop is constant and indicated by ω



This system has a single degree of freedom, θ

$$T = \frac{1}{2} m \rho^2 \omega^2 + \frac{1}{2} m (R\dot{\vartheta})^2$$

$$= \frac{m}{2} \left(R^2 \sin^2\vartheta\, \omega^2 + R^2 \dot{\vartheta}^2 \right)$$

$$= \frac{m}{2} R^2 \left(\sin^2\vartheta\, \omega^2 + \dot{\vartheta}^2 \right)$$

$$U = m g R (1 - \cos\vartheta)$$

Therefore the Lagrangian is

$$\boxed{\mathcal{L} = \frac{mR^2}{2} \left(\sin^2\vartheta\, \omega^2 + \dot{\vartheta}^2 \right) - mgR(1-\cos\vartheta)}$$

The equation of motion for the angle θ is

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m R^2 \omega^2 \sin\vartheta \cos\vartheta - mgR\sin\vartheta$$
$$- m R^2 \ddot{\vartheta} = 0$$

$$\boxed{\ddot{\vartheta} = \left(\omega^2 \cos\vartheta - \frac{g}{R}\right)\sin\vartheta}$$

The equation cannot be solved analytically

Equilibrium points

However, one can use the equation of motion to find out what are the equilibrium points, namely the values of θ for which, if the bead is placed there with zero generalized velocity, the bead will remain in place. Clearly the equilibrium points will be the points where the generalized acceleration will be zero acceleration

$$\ddot{\vartheta} = 0 \longrightarrow \left(\omega^2 \cos\vartheta - \frac{g}{R}\right)\sin\vartheta = 0$$

The equation above has four solutions

1) $\vartheta = 0$
2) $\vartheta = \pi$

$\left.\right\} \rightarrow \sin\vartheta = 0$

3) $\vartheta = \arccos\left(+\frac{g}{R\omega^2}\right)$
4) $\vartheta = -\arccos\left(+\frac{g}{R\omega^2}\right)$

$\left.\right\} \rightarrow \omega^2\cos\vartheta - \frac{g}{R} = 0$

but one needs $\frac{g}{R\omega^2} < 1$

Now we want to determine if a given equilibrium point is stable or unstable. An equilibrium point is called stable if the bead returns to the equilibrium point if it is moved away from it by an infinitesimal amount. An equilibrium point is called unstable if the bead moves away from it if displaced by an infinitesimal amount from the equilibrium point. In order to determine the nature of the equilibrium point it is necessary to look at the acceleration in theta.

Point 1

if $\vartheta \simeq 0 \longrightarrow \cos\vartheta \simeq 1 , \sin\vartheta \simeq \vartheta$

$$\ddot{\vartheta} = \left(\omega^2 - \frac{g}{R}\right)\vartheta$$

if $\omega < \sqrt{\frac{g}{R}}$ this equilibrium point is stable because $\ddot{\vartheta} < 0$

if $\omega > \sqrt{\frac{g}{R}}$ this equilibrium point is unstable

Point 2

if $\vartheta \simeq \pi$, $\cos(\pi - \alpha) \simeq -1$, $\sin(\pi - \alpha) = \sin\alpha \simeq \alpha$

$$\ddot{\alpha} = + \left(\omega^2 + \frac{g}{R}\right)\alpha \qquad \ddot{\vartheta} = -\ddot{\alpha}$$

Since $\ddot{\vartheta}$ is always negative, the bead will accelerate toward smaller values of ϑ, so away from $\vartheta \simeq \pi \longrightarrow$ This equilibrium point is always unstable

Point 3

Point 3 and point 4 are equilibrium points only when point 1 is unstable, i.e. when

$$\omega > \sqrt{\frac{g}{R}}$$

Consider now an angle $\theta < \pi/2$ such that

$$\omega^2 \cos\vartheta_0 - \frac{g}{R} = 0$$

Then let's set

$$\vartheta = \vartheta_0 + \delta\vartheta$$

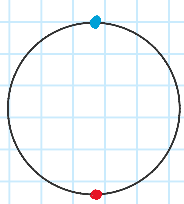
$$\delta\ddot{\vartheta} = \left(\omega^2 \cos\vartheta - \frac{g}{R}\right)\sin\vartheta \simeq -\omega^2 \sin^2\vartheta_0\, \delta\vartheta$$

$$\left.\begin{array}{l} \text{if } \delta\vartheta > 0 \longrightarrow \delta\ddot{\vartheta} < 0 \\ \text{if } \delta\vartheta < 0 \longrightarrow \delta\ddot{\vartheta} > 0 \end{array}\right\} \text{ point 3 is stable}$$

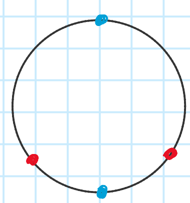
The same considerations apply to point 4.

To summarize

- unstable
- stable



$$\omega < \sqrt{\frac{g}{R}}$$



$$\omega > \sqrt{\frac{g}{R}}$$

Oscillations near equilibrium

The equation of motion can be simplified if one considers small oscillation around equilibrium. For point 1 one finds

$$\vartheta \ll 1 \qquad \ddot{\vartheta} = -\left(\frac{g}{R} - \omega^2\right)\vartheta$$

$$\ddot{\vartheta} = -\Omega^2 \vartheta \qquad \Omega = \sqrt{\frac{g}{R} - \omega^2}$$

$$\text{if } \frac{g}{R} > \omega^2 \longrightarrow \Omega > 0 \longrightarrow$$ SIMPLE HARMONIC MOTION

$$\vartheta(t) = A \cos \Omega t$$

for $\omega^2 > \frac{g}{R} \rightarrow \Omega = i\Omega_R \qquad \Omega_R \in \mathbb{R}^+$

then $\qquad \cos(i\Omega_R t) = \underbrace{\cosh(\Omega_R t)}_{\text{grows with time}}$

For point 3 one can define

$$\omega^2 \cos\vartheta_0 - \frac{g}{R} \equiv 0$$

then set $\qquad \vartheta = \vartheta_0 + \delta\vartheta$

$$\cos(\vartheta_0 + \delta\vartheta) = \cos\vartheta_0 - \delta\vartheta \sin\vartheta_0$$

$$\sin(\vartheta_0 + \delta\vartheta) = \sin\vartheta_0 + \delta\vartheta \cos\vartheta_0$$

Therefore

$$\ddot{\vartheta} = \left[\omega^2 \cos(\vartheta_0 + \delta\vartheta) - \frac{g}{R}\right] \sin(\vartheta_0 + \delta\vartheta)$$

$$= -\omega^2 \sin^2\vartheta_0\, \delta\vartheta + \ldots$$

but $\qquad \ddot{\vartheta} = \delta\ddot{\vartheta}$

$$\delta\ddot{\vartheta} = -\underbrace{\omega^2 \sin^2\vartheta_0}_{\equiv \Omega'^2} \delta\vartheta$$

SIMPLE HARMONIC MOTION

$$(\Omega')^2 = \omega^2 \sin^2\vartheta_0 = \omega^2(1 - \cos^2\vartheta_0)$$

$$= \omega^2 \left(1 - \frac{g^2}{R^2 \omega^4}\right) = \omega^2 - \frac{g}{R\omega^2}$$

$$\Omega' = \sqrt{\omega^2 - \frac{g}{R\omega^2}}$$

FREQUENCY OF THE OSCILLATION

Observation

Notice that in this case θ is a coordinate with respect to a non-inertial frame (the rotating hoop). However, what matters is that the Lagrangian is written in an inertial frame (as we did, considering an observer at rest with respect to the ground and not an observer rotating together with the hoop). This is sufficient to obtain Lagrange's equation which correctly describe the physics of the problem.