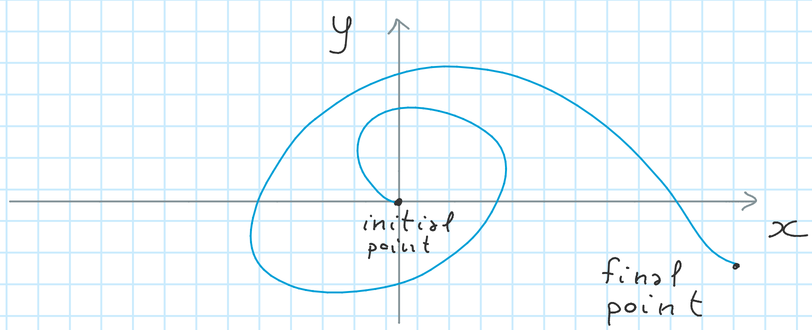
Functionals depending on more than one variable

Tuesday, August 20, 2019 2:35 PM

For applications in mechanics it is important to consider the case in which the argument of the integral for which one needs to find the stationary points depends on more than one variable. In order to study this (simple) generalization it is convenient to reconsider the problem of finding the shortest distance between two points. By looking back at what we did we can see that in order to find the shortest path among two points we considered only paths for which we could write y as a function of x. However, this excludes paths that are looping on themselves (it is obvious that these paths cannot contain the shortest distance between two points, but still, mathematically, we did not prove it). Let's then consider arbitrary paths in which one can parameterize both the x and the y coordinate in terms of a third parameter u.



$$x \equiv x(u) \qquad y \equiv y(u)$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

$$= \sqrt{(x')^2 + (y')^2}\, du$$

Where the prime now indicates a derivative with respect to u. The length of the curve can be written as

$$L = \int_{u_1}^{u_2} \sqrt{\left(x'(u)\right)^2 + \left(y'(u)\right)^2}\, du$$

More in general, we are interested in looking for the stationary points for an integral of the form

$$S = \int_{u_1}^{u_2} f\left[x(u), y(u), x'(u), y'(u)\right] du$$

One can generalize the one variable case by considering a small variation of both x and y around the path that corresponds to a maximum or a minimum of S.

$$X(u) = x(u) + \alpha\,\xi(u) \qquad Y(u) = y(u) + \beta\,\eta(u)$$

$$\xi(u_1) = \eta(u_1) = \xi(u_2) = \eta(u_2) = 0$$

S is now a function of α and β. A necessary condition to have a stationary point will be

$$\left.\frac{\partial S}{\partial \alpha}\right|_{\substack{\alpha=0\\ \beta=0}} = 0 \quad \text{and} \quad \left.\frac{\partial S}{\partial \beta}\right|_{\substack{\alpha=0\\ \beta=0}} = 0$$

$$\frac{\partial S}{\partial \alpha} = \int_{u_1}^{u_2} \left[\xi(u)\frac{\partial f}{\partial x} + \xi'(u)\frac{\partial f}{\partial x'} \right] du \quad \to \text{by parts}$$

$$= \int_{u_1}^{u_2} \left[\xi(u)\frac{\partial f}{\partial x} - \xi(u)\frac{d}{du}\frac{\partial f}{\partial x'} \right] du + \underbrace{\cancel{\left[\xi(u)\frac{\partial f}{\partial x'} \right]_{u_1}^{u_2}}}_{0}$$

$$= 0$$

$$\to \boxed{\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0}$$

similarly

$$\frac{\partial S}{\partial \beta} = 0 \longrightarrow \boxed{\frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} = 0}$$

One finds then two Euler Lagrange equations.
Let's check that for the case of the shortest path between two points we recover the expected answer (I.e. a straight line)

$$L = \int_{u_1}^{u_2} \sqrt{(x'(u))^2 + (y'(u))^2}\, du$$

$$f = \sqrt{(x'(u))^2 + (y'(u))^2}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{(x')^2 + (y')^2}} \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{(x')^2 + (y')^2}}$$

$$\frac{d}{du}\frac{\partial f}{\partial x'} = 0 \longrightarrow \frac{x'}{\sqrt{(x')^2 + (y')^2}} = K_1$$

constant

$$\frac{d}{du}\frac{\partial f}{\partial y'} = 0 \longrightarrow \frac{y'}{\sqrt{(x')^2 + (y')^2}} = K_2$$

$$\frac{y'}{x'} = \frac{dy}{dx} = \frac{K_2}{K_1} \equiv K \longrightarrow \boxed{y = Kx + y_0}$$