Euler Lagrange equation

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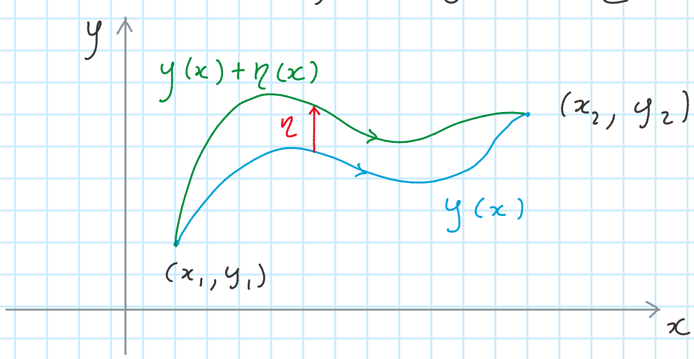
Here we want to solve in general terms the problem of choosing a path that minimizes the integrals defined in the two examples discussed above, namely, what is the shortest distance between two points in a plane and what is the shortest time light will take to go from a given initial point to a given final point.

Both problems require to minimize an integral of the general form

$$S = \int_{x_1}^{x_2} f\left[y(x), y'(x), x\right] dx$$

As a first step and unavoidable step, we try to figure out for which paths y(x) S has a stationary value. Let's suppose that a certain (unknown) y(x) corresponds to the minimum value of S. If we deform slightly the path y(x), then S will have a larger value than the one it has for the curve y(x) that corresponds to the minimum.

$$y(x) \longrightarrow S_{min} \quad , \quad y(x) + \eta(x) \longrightarrow S > S_{min}$$



The endpoints of the curve, and of the integral, are however fixed

$$\eta(x_1) = \eta(x_2) = 0$$

One can actually parameterize a family of curves that are "close" to y(x) but do not coincide with y(x) by introducing the parameter α

$$Y(x) \equiv y(x) + \alpha\, \eta(x)$$

If one inserts Y in the integral S, also the latter will depend on α

$$S(\alpha) \equiv \int_{x_1}^{x_2} f(Y, Y', x)\, dx$$

$$= \int_{x_1}^{x_2} f(y+\alpha\eta, y'+\alpha\eta', x)\, dx$$

S is now a regular function of α. Consequently, it will have a stationary point when its derivative vanishes

$$\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha}\, dx = \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}\right) dx = 0$$

Now apply integration by parts to the second term in the round bracket

$$\int_{x_1}^{x_2} \left(\frac{d}{dx}\eta(x)\right) \frac{\partial f}{\partial y'}\, dx = \underbrace{\left[\eta(x) \frac{\partial f}{\partial y'}\right]_{x_1}^{x_2}}_{\substack{=0 \ \text{since} \\ \eta(x_1)=\eta(x_2)=0}} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

Therefore

$$\left.\frac{d}{d\alpha} S\right|_{\alpha=0} = \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} - \eta \frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx = 0$$

$$\to \int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx = 0$$

Since we are dealing with continuous functions, and we are free to choose an arbitrary function η, the integral will be zero only if

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$

EULER LAGRANGE EQUATION

Let's consider again the last logical step. Can one have

$$\int_{x_1}^{x_2} \eta(x)\, F(x)\, dx = 0$$

for arbitrary η if F is not equal to zero? Remember the assumption that we are dealing with smooth continuous functions. If F is not zero for every x, we can choose a η that has the same sign of F in each point x. Therefore, the integrand will be positive in each point where F is different from zero. This implies that the integral will only receive positive contributions, so it cannot be zero. We reached a contradiction. Consequently our assumption that the integral is zero is incompatible with the assumption that F is not zero everywhere. Consequently

$$\int_{x_1}^{x_2} \eta(x)\, F(x)\, dx = 0 \quad \forall \eta \longrightarrow \quad F(x) = 0$$

Consequently, by requiring that y(x) satisfies the Euler Lagrange equation we can find the solution of the problems which we are trying to solve. As a first application let's go back to the case of the problem of finding the shortest path between two points

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx \longrightarrow f(x) \equiv \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = \frac{1}{2} \frac{2y'}{\sqrt{1 + (y')^2}} = \frac{y'}{\sqrt{1 + (y')^2}}$$

$$\frac{d}{dx} \frac{\partial f}{\partial y'} = \frac{d}{dx} \left(\frac{y'}{\sqrt{1 + (y')^2}} \right) = 0$$

$$\hookrightarrow \qquad y' = C \sqrt{1 + (y')^2} \longrightarrow (y')^2 = C^2 (1 + (y')^2)$$

$$(y')^2 (1 - C^2) = C^2 \longrightarrow (y')^2 = \frac{C^2}{1 - C^2}$$

$$\hookrightarrow \qquad y' \equiv K \leftarrow \text{constant}$$

$$\boxed{y = kx + x_0}$$

Impose the boundary conditions

$$y(x_1) = y_1 \longrightarrow y_1 = k x_1 + x_0 \quad , \quad y(x_2) = y_2 \longrightarrow y_2 = k x_2 + x_0$$

$$y_2 - y_1 = k(x_2 - x_1) \longrightarrow k = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y_1 = \frac{y_2 - y_1}{x_2 - x_1} x_1 + x_0 \longrightarrow x_0 = \frac{y_1 x_2 - \cancel{y_1 x_1} - y_2 x_1 + \cancel{y_1 x_1}}{x_2 - x_1}$$

$$x_0 = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$

$$\boxed{y = \frac{y_2 - y_1}{x_2 - x_1} x + \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}}$$

Check

$$x \to x_2 \qquad y = \frac{y_2 x_2 - \cancel{y_1 x_2} + \cancel{y_1 x_2} - y_2 x_1}{x_2 - x_1} = y_2 \quad \checkmark$$