

# Driven damped oscillator

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In order to keep an oscillator in motion in presence of a damping force, it is necessary to supply a driving force. For example, the push that is supplied by a person on the ground to keep a swing in motion is an example of a driving force. The equation of motion satisfied by a damped-driven oscillator can then be written as

$$m \ddot{x} + \underbrace{b \dot{x}}_{\text{damping force}} + kx = \underbrace{F(t)}_{\text{driving force}}$$

The equation can be rewritten by dividing each term by the mass, so that the coefficient of the second derivative is normalized to 1.

$$\ddot{x} + \underbrace{\frac{b}{m} \dot{x}}_{\equiv 2\beta} + \underbrace{\frac{k}{m} x}_{\omega_0^2} = \underbrace{\frac{F(t)}{m}}_{f(t)}$$

$$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = f(t)$$

This is an inhomogeneous linear differential equation. Indeed one can define a linear differential operator  $D$  as follows

$$D \equiv \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2$$

The operator is linear because, given two solutions of the homogeneous equation (damped but not driven oscillator) any linear combination of the two solutions is still a solution of the homogeneous equation

$$\text{if } D x_1 = 0 \text{ and } D x_2 = 0 \rightarrow D(a x_1 + b x_2) = 0$$

The equation that we want to solve is an inhomogeneous equation because the operator  $D$  applied to the function  $x$  is not equal to zero

$$D x = \underbrace{f(t)}_{\text{inhomogeneous term}}$$

## Solution of the inhomogeneous equation

If one manages to find one solution of the inhomogeneous equation and a general solution of the homogeneous equation, then one can find all solutions of the inhomogeneous equation. In fact

$$\begin{aligned} \text{if } D x_p &= f(t) & x_p &\rightarrow \text{"particular solution,"} \\ \text{and } D x_h &= 0 & x_h &\rightarrow \text{"homogeneous solution,"} \\ \text{then } D(x_p + x_h) &= D x_p + \underbrace{D x_h}_{=0} = f(t) \end{aligned}$$

The sum of the particular and homogeneous solutions depends on two integration constants, therefore it is already the most general solution of a second order differential equation (not a rigorous proof of course!)

## Case of sinusoidal driving force

It is convenient to consider the case in which the driving force is a sinusoidal function. This has two purposes: A sinusoidal function reasonably approximates most periodic functions. Secondly, any driving force can be written by means of Fourier analysis as a superposition of sinusoidal forces. The differential equation that one needs to solve is then

$$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

driving  
frequency  
↓

observe : in general,  $\omega \neq \omega_0$

In order to solve the equation, it is convenient to combine the equation above with the related equation

$$\ddot{y} + 2\beta \dot{y} + \omega_0^2 y = f_0 \sin(\omega t)$$

One can then define a complex function  $z(t)$  that satisfies a differential equation where one finds an exponential rather than a sinusoidal function.

$$z(t) \equiv x(t) + i y(t)$$

$$(\ddot{x} + i\ddot{y}) + 2\beta(\dot{x} + i\dot{y}) + \omega_0^2(x + iy) = f_0(\cos \omega t + i \sin \omega t)$$

$$\hookrightarrow \ddot{z} + 2\beta \dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

Since the derivative of an exponential is the exponential itself, one can try the Ansatz

$$z(t) = C e^{i\omega t}$$

$$(-\omega^2 e^{+i\omega t} + 2i\beta\omega e^{i\omega t} + \omega_0^2 e^{i\omega t}) C = f_0 e^{i\omega t}$$

$$C = \frac{f_0}{(\omega_0^2 - \omega^2) + 2i\beta\omega}$$

It is convenient to write the constant  $C$  in terms of a real amplitude and a phase.

$$C \equiv A e^{-i\delta} = \frac{f_0}{(\omega_0^2 - \omega^2) + 2i\beta\omega}$$

$$A^2 = C^* C = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \rightarrow A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$e^{-i\delta} = \frac{f_0}{A} \frac{1}{(\omega_0^2 - \omega^2) + 2i\beta\omega}$$

$$f_0 e^{i\delta} = A [(\omega_0^2 - \omega^2) + 2i\beta\omega] \rightarrow \begin{aligned} f_0 \cos \delta &= A(\omega_0^2 - \omega^2) \\ f_0 \sin \delta &= 2A\beta\omega \end{aligned}$$

$$\tan \delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2} \rightarrow \delta = \arctan \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$



If one instead varies  $\omega$  keeping the natural frequency fixed, the maximum of the amplitude is reached for

$$\frac{d}{d\omega} \left( (\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2 \right) = 0 \quad \text{minimum for the denominator}$$

$$-2(\omega_0^2 - \omega^2)2\omega + 8\beta^2\omega = 0$$

$$4\omega \left[ 2\beta^2 - (\omega_0^2 - \omega^2) \right] = 0 \rightarrow \omega_0^2 - \omega^2 = 2\beta^2$$

$\omega=0$   $\rightarrow$  local minimum of the amplitude  
 $= 0 \rightarrow$  maximum of the amplitude

$$\omega = \sqrt{\omega_0^2 - 2\beta^2}$$

$\omega \approx \omega_0$  for  $\beta \rightarrow 0$

Indeed by studying the second derivative of the denominator one finds

$$\frac{d^2}{d\omega^2} \left( (\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2 \right) = 0$$

$$\frac{d}{d\omega} 4\omega \left[ 2\beta^2 - (\omega_0^2 - \omega^2) \right] =$$

$$4 \left[ 2\beta^2 - (\omega_0^2 - \omega^2) \right] + 8\omega^2 =$$

$$4 \left[ 2\beta^2 - \omega_0^2 + \omega^2 + 2\omega^2 \right] = 4 \left( 2\beta^2 - \omega_0^2 + 3\omega^2 \right)$$

at  $\omega = 0$  the second derivative is

$$\frac{d^2}{d\omega^2} (\dots) \Big|_{\omega=0} = 4(2\beta^2 - \omega_0^2) \quad \begin{array}{l} > 0 \text{ if } \beta^2 > \frac{\omega_0^2}{2} \\ < 0 \text{ if } \beta^2 < \frac{\omega_0^2}{2} \end{array}$$

so if  $\omega_0^2 - 2\beta^2 > 0$ ,  $\omega = 0$  corresponds to a maximum of the denominator and to a minimum of the squared amplitude.

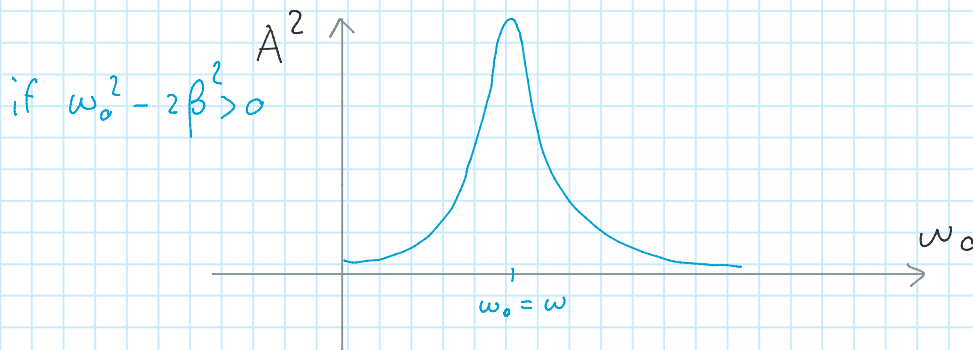
At  $\omega = \sqrt{\omega_0^2 - 2\beta^2}$  one finds

$$\begin{aligned}\frac{d^2}{d\omega^2}(\dots) &= 4 \left[ 2\beta^2 - \omega_0^2 + 3\omega_0^2 - 6\beta^2 \right] \\ &= 4 \left( 2\omega_0^2 - 4\beta^2 \right) \\ &= 8 \left( \omega_0^2 - 2\beta^2 \right)\end{aligned}$$

The above is  $> 0$  for  $\omega_0^2 - 2\beta^2 > 0$

Therefore  $\omega = \sqrt{\omega_0^2 - 2\beta^2}$  is a minimum of the denominator and a maximum of the amplitude, as declared above.

The situation can be illustrated as follows



Width of the resonance

In order to get a sense of how wide or narrow a resonance is, it is instructive to study the width of the amplitude squared half way to the maximum: This is called the full width at half maximum or FWHM

For a small damping factor  $\beta$  the maximum is obtained for  $\omega$  equal to the natural frequency, therefore

$$A_{\max}^2 \approx \frac{f_0^2}{4\beta^2 \omega_0^2} \quad \longrightarrow$$

$$\frac{1}{2} A_{\max}^2 = \frac{f_0^2}{8\beta^2\omega_0^2} = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

$$\hookrightarrow 8\beta^2\omega_0^2 = (\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2$$

$$\omega^4 - 2\omega^2\omega_0^2 + \omega_0^4 + 4\beta^2(\underbrace{\omega^2 - 2\omega_0^2}_{\simeq -\omega_0^2}) = 0$$

$$\omega^4 - 2\omega^2\omega_0^2 + \omega_0^4 - 4\beta^2\omega_0^2 = 0$$

$$\omega^2 = \omega_0^2 \pm \sqrt{\cancel{\omega_0^4} - \cancel{\omega_0^4} + 4\beta^2\omega_0^2}$$

$$= \omega_0^2 \pm 2\beta\omega_0$$

$$\omega = \sqrt{\omega_0^2 \pm 2\beta\omega_0} \simeq \sqrt{(\omega_0 \pm \beta)^2} \simeq \omega_0 \pm \beta$$

Consequently, the FWHM is approximately  $2\beta$ . The smallest the damping factor, the narrower the resonance.