Oscillations at the bottom of a half pipe

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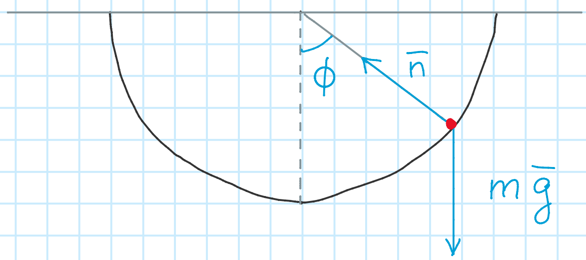
As an application of Newton second law in polar coordinates we consider the motion of a point like marble in a half pipe with circular section



The distance of the marble from the center of the circle is constant, therefore the Newton's second law simplifies because the derivatives of the radial coordinate are zero

$$-mR\dot{\phi}^2 = F_r \qquad mR\ddot{\phi} = F_\phi$$

One can now plug in

$$F_r = -n + mg\cos\phi \qquad F_\phi = -mg\sin\phi$$

$$\boxed{+mR\dot{\phi}^2 = n - mg\cos\phi} \qquad \boxed{mR\ddot{\phi} = -mg\sin\phi}$$

The equation that allows one to determine phi as a function of time is the second one . The first equation allows one to find out n as a function of time (not relevant for our purposes)

$$\boxed{\ddot{\phi} = -\frac{g}{R}\sin\phi}$$

This equation can be solved in terms of elliptic functions. However if we consider only small oscillations near the bottom of the pipe the equation becomes more familiar and simpler to solve if ϕ is small $\longrightarrow$ $\sin\phi \simeq \phi$

$$\boxed{\ddot{\phi} = -\frac{g}{R}\,\phi}$$

SIMPLE HARMONIC MOTION

The most general solution for this differential equation is

$$\boxed{\phi(t) = A\cos(\omega t) + B\sin(\omega t)}$$

with $\omega = \sqrt{\frac{g}{R}}$

This general solution depends on two unspecified constants, that should be fixed by specifying the initial position and velocity of the marble. Since we are solving a second order differential equation we expect to deal with a general solution that depends on two unspecified constants. Let's assume that the marble is released at rest at certain initial angle with respect to the vertical direction.

$$\phi(0) = A \equiv \phi_0$$

$$\dot{\phi}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$\dot{\phi}(0) = B\omega \equiv 0$$ (released from rest)

$$\phi(t) = \phi_0\cos(\omega t)$$

The marble goes back to the initial position after a time tau such that

$$\omega\tau = 2\pi \qquad \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}$$

PERIOD OF THE OSCILLATION