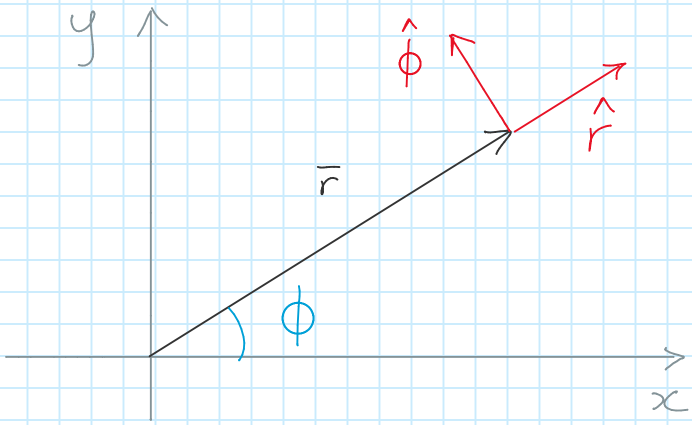
2nd law in polar coordinates

Tuesday, June 4, 2019 4:12 PM

It is often useful to employ systems of coordinates which are not cartesian. For example, some two dimensional problems are conveniently treated in polar coordinates



$$\boxed{\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ r &= \sqrt{x^2+y^2} \\ \phi &= \arctan\frac{y}{x} \end{aligned}}$$

unit vectors: $|\hat{r}|=1 \quad |\hat{\phi}|=1$

The net force can have a component along r and a component along phi, while the position only has a component along r:

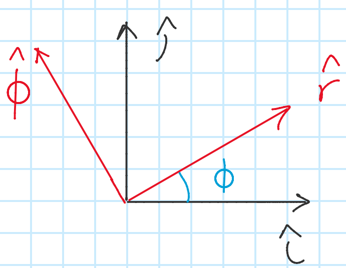
$$\bar{F} = F_r\,\hat{r} + F_\phi\,\hat{\phi} \qquad\qquad \bar{r} = r\,\hat{r}$$

In order to write Newton's law in polar coordinates we still need to take time derivatives of the vector r and then write the result in polar coordinates

$$\bar{r} = x\,\hat{\imath} + y\,\hat{\jmath} = r\cos\phi\,\hat{\imath} + r\sin\phi\,\hat{\jmath}$$

$$\dot{\bar{r}} = \left[\dot{r}\cos\phi - r(\sin\phi)\dot{\phi}\right]\hat{\imath} + \left[\dot{r}\sin\phi + r(\cos\phi)\dot{\phi}\right]\hat{\jmath}$$

Now observe that



$$\hat{\imath} = \cos\phi\,\hat{r} - \sin\phi\,\hat{\phi}$$
$$\hat{\jmath} = \sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}$$

Therefore

$$\dot{\bar{r}} = \left(\dot r \cos\phi - r\dot\phi \sin\phi\right)\left(\underbrace{\cos\phi\,\hat r - \sin\phi\,\hat\phi}_{\hat\imath}\right) + \left(\dot r \sin\phi + r\dot\phi\cos\phi\right)\left(\underbrace{\sin\phi\,\hat r + \cos\phi\,\hat\phi}_{\hat\jmath}\right)$$

$$\dot{\bar{r}} = \dot r\cos^2\phi\,\hat r - \cancel{r\dot\phi\sin\phi\cos\phi\,\hat r} - \cancel{\dot r\cos\phi\sin\phi\,\hat\phi} + r\dot\phi\sin^2\phi\,\hat\phi + \dot r\sin^2\phi\,\hat r + \cancel{r\dot\phi\cos\phi\sin\phi\,\hat r} + \cancel{\dot r\sin\phi\cos\phi\,\hat\phi} + r\dot\phi\cos^2\phi\,\hat\phi$$

$$\boxed{\dot{\bar{r}} = \dot r\,\hat r + r\dot\phi\,\hat\phi}$$

Often the following notation is used

$\dot\phi \equiv \omega$ angular velocity

$$\bar v = \dot{\bar r} = v_r\,\hat r + v_\phi\,\hat\phi \qquad \begin{cases} v_r = \dot r & \text{radial velocity} \\ v_\phi = r\dot\phi = r\omega & \text{tangential velocity} \end{cases}$$

Observe the following

$$\dot{\bar r} = \frac{d\bar r}{dt} = \frac{d}{dt}(r\,\hat r) = \dot r\,\hat r + r\frac{d\hat r}{dt}$$

Compare with the result above

$$\dot{\bar r} = \dot r\,\hat r + r\dot\phi\,\hat\phi = \dot r\,\hat r + r\frac{d\hat r}{dt} \qquad \boxed{\frac{d\hat r}{dt} = \dot\phi\,\hat\phi}$$

One concludes then that the unit vectors in polar coordinates are time dependent (indeed their direction changes in time, something that does not happen in cartesian coordinates)

One still needs to calculate the second derivative

$$\ddot{\bar r} = \frac{d}{dt}\dot{\bar r} = \frac{d}{dt}\left(\dot r\cos\phi - r\dot\phi\sin\phi\right)\hat\imath + \frac{d}{dt}\left(\dot r\sin\phi + r\dot\phi\cos\phi\right)\hat\jmath$$

$$= \left(\ddot r\cos\phi - \dot r\dot\phi\sin\phi - \dot r\dot\phi\sin\phi - r\ddot\phi\sin\phi - r(\dot\phi)^2\cos\phi\right)\hat\imath$$

$$+ \left(\ddot r\sin\phi + \dot r\dot\phi\cos\phi + \dot r\dot\phi\cos\phi + r\ddot\phi\cos\phi - r(\dot\phi)^2\sin\phi\right)\hat\jmath$$

Now rewrite the cartesian unit vectors as functions of the polar unit vectors, as we did above

shorthand notation $\quad \cos\phi \equiv c \quad \sin\phi \equiv s$

$$\ddot{\vec r} = \left(\ddot r c - \dot r\dot\phi s - \dot r\dot\phi s - r\ddot\phi s - r(\dot\phi)^2 c\right)\left(c\hat r - s\hat\phi\right)$$

$$+\left(\ddot r s + \dot r\dot\phi c + \dot r\dot\phi c + r\ddot\phi c - r(\dot\phi)^2 s\right)\left(s\hat r + c\hat\phi\right)$$

$$= \left(\ddot r c^2 - \cancel{2\dot r\dot\phi sc} - \cancel{r\ddot\phi sc} - r(\dot\phi)^2 c^2\right)\hat r$$

$$-\left(\cancel{\ddot r cs} - 2\dot r\dot\phi s^2 - r\ddot\phi s^2 - \cancel{r(\dot\phi)^2 cs}\right)\hat\phi$$

$$+\left(\ddot r s^2 + \cancel{2\dot r\dot\phi cs} + \cancel{r\ddot\phi sc} - r(\dot\phi)^2 s^2\right)\hat r$$

$$+\left(\cancel{\ddot r sc} + 2\dot r\dot\phi c^2 + r\ddot\phi c^2 - \cancel{r(\dot\phi)^2 sc}\right)\hat\phi$$

$$\boxed{\ddot{\vec r} = \left(\ddot r - r(\dot\phi)^2\right)\hat r + \left(2\dot r\dot\phi + r\ddot\phi\right)\hat\phi}$$

The complicated expression above looks partially better if one considers the case in which the radial distance does not change

$$\ddot{\vec r} = \underbrace{-r\dot\phi^2\,\hat r}_{\text{centripetal acceleration}} + \underbrace{r\ddot\phi\,\hat\phi}_{\text{tangential acceleration}} \qquad \ddot\phi = \alpha \quad \text{angular acceleration}$$

In addition from the above one can find out the derivative of the unit vector in the phi direction

$$\ddot{\vec r} = \frac{d\dot{\vec r}}{dt} = \frac{d}{dt}\left(\dot r\hat r + r\dot\phi\hat\phi\right) = \ddot r\hat r + \dot r\frac{d\hat r}{dt} + \dot r\dot\phi\hat\phi + r\ddot\phi\hat\phi + r\dot\phi\frac{d\hat\phi}{dt}$$

$$= \ddot r\hat r + \dot r\dot\phi\hat\phi + \dot r\dot\phi\hat\phi + r\ddot\phi\hat\phi + r\dot\phi\frac{d\hat\phi}{dt}$$

$$= \ddot{r}\hat{r} + 2\dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\frac{d\hat{\phi}}{dt}$$

Compare with the previous result

$$\ddot{\bar{r}} = \left(\ddot{r} - r(\dot{\phi})^2\right)\hat{r} + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\hat{\phi}$$

$$= \ddot{r}\hat{r} + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\hat{\phi} + r\dot{\phi}\frac{d\hat{\phi}}{dt}$$

$$\boxed{\frac{d\hat{\phi}}{dt} = -\dot{\phi}\ \hat{r}}$$

One can finally write down Newton's second law in polar coordinates

$$m\ddot{\bar{r}} = \bar{F}$$

$$\boxed{\begin{aligned} m\left(\ddot{r} - r(\dot{\phi})^2\right) &= F_r \\ m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) &= F_\phi \end{aligned}}$$

Lagrangian mechanics will help in obtaining the differential equations we need to solve in a simpler way also for coordinates that are not cartesian.