Lagrange equations for holonomic systems

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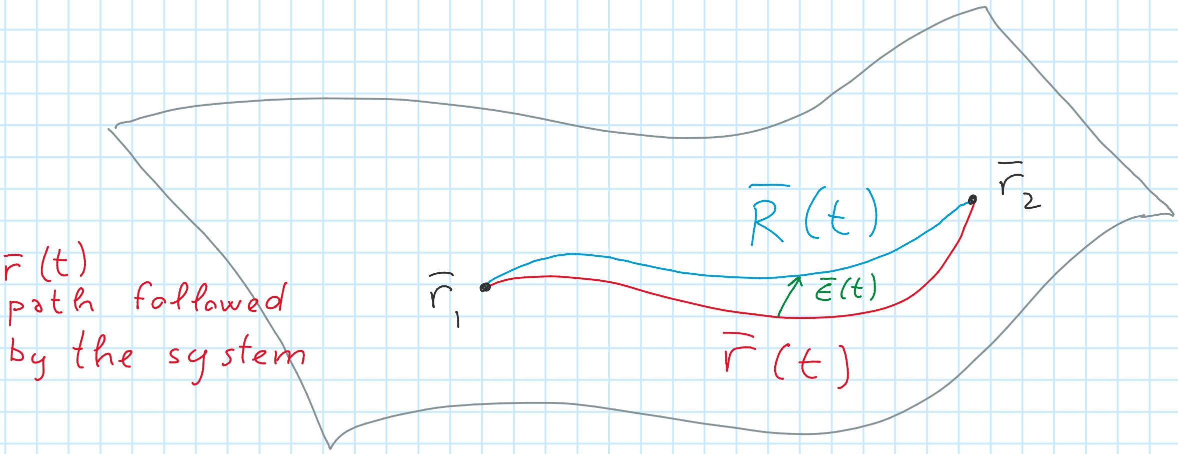
The goal here is to prove that a system subject to holonomic constraints satisfies Lagrange's equations of motion. In order to keep the notation simple, consider a single particle constrained to move on a surface (which can or cannot be flat, we do not make assumptions on this).

There are two kinds of forces acting on the particle: The forces of constraint (in this case the normal force applied by the surface on which the particle is moving) and the non-constraint forces. We assume that the non constraint forces are conservatives. No assumptions are made on the constraint forces. In our case the normal force does not do work. A more interesting example is the rigid body, where the constraint forces are the interatomic forces that keep the body together. Those forces might be non conservative, but the discussion below is still valid, since we are only requiring that the non-constraint forces are conservative.

In summary we assume

$$\bar{F}_{\substack{\text{non}\\ \text{const}}} = -\nabla U(\bar{r},t) \qquad\qquad \mathcal{L} = T - \overset{\substack{\text{potential for}\\ \text{non-constraint}\\ \text{forces}}}{\boxed{U}}$$

Consider the path taken by the particle on the surface between two points and then consider a path close to the one followed by the system



$$\bar{r}(t_1) \equiv \bar{r}_1 \qquad \bar{r}(t_2) \equiv \bar{r}_2 \qquad\qquad \bar{R}(t) = \bar{r}(t) + \bar{\varepsilon}(t)$$

One can then define the action along the two paths and prove that in the vicinity of the path followed by the system, the action is stationary.

$$S_0 = \int_{t_1}^{t_2} \mathcal{L}\left(\bar{r}(t), \dot{\bar{r}}(t), t\right) dt$$

$$S = \int_{t_1}^{t_2} \mathcal{L}\left(\bar{R}(t), \dot{\bar{R}}(t), t\right) dt$$

want to prove that $\boxed{\delta S = S - S_0 = \mathcal{O}(\epsilon^2)}$

Proof:

$$\delta \mathcal{L} = \mathcal{L}(\bar{R}, \dot{\bar{R}}, t) - \mathcal{L}(\bar{r}, \dot{\bar{r}}, t)$$

Remember that

$$\mathcal{L}(\bar{r}, \dot{\bar{r}}, t) = \frac{1}{2} m \dot{\bar{r}}^2 - U(\bar{r}, t) \quad , \quad \bar{R} = \bar{r} + \bar{\epsilon}$$

$$\delta \mathcal{L} = \frac{1}{2} m \left[(\dot{\bar{r}} + \dot{\bar{\epsilon}})^2 - \dot{\bar{r}}^2 \right] - \left[U(\bar{r} + \bar{\epsilon}, t) - U(\bar{r}, t) \right]$$

$$= m \dot{\bar{r}} \cdot \dot{\bar{\epsilon}} - \bar{\epsilon} \cdot \nabla U + \mathcal{O}(\epsilon^2)$$

Therefore

$$\delta S = \int_{t_1}^{t_2} \delta \mathcal{L} \, dt = \int_{t_1}^{t_2} \left[m \dot{\bar{r}} \cdot \dot{\bar{\epsilon}} - \bar{\epsilon} \cdot \nabla U \right] dt + \mathcal{O}(\epsilon^2)$$

Now observe that

$$\int_{t_1}^{t_2} m \dot{\bar{r}} \cdot \dot{\bar{\epsilon}} \, dt = \cancelto{0}{\left[m \dot{\bar{r}} \cdot \bar{\epsilon} \right]_{t_1}^{t_2}} - \int_{t_1}^{t_2} m \ddot{\bar{r}} \cdot \bar{\epsilon} \, dt$$

since $\bar{\epsilon}(t_1) = \bar{\epsilon}(t_2) = 0$

The variation of the action can then be written as

$$\delta S = \int_{t_1}^{t_2} \delta \mathcal{L}\, dt = -\int_{t_1}^{t_2} [m\ddot{\bar{r}} + \nabla U]\cdot \bar{\epsilon}\, dt$$

Since r is the path followed by the system, Newton's second law applies

$$m\ddot{\bar{r}} = \bar{F}_{const} + \bar{F}_{\substack{non\\ const}} = \bar{F}_{const} - \nabla U$$

$$m\ddot{\bar{r}} + \nabla U = \bar{F}_{const}$$

$$\delta S = -\int_{t_1}^{t_2} \underbrace{\bar{\epsilon}\cdot \bar{F}_{const}}_{=0}\, dt + \mathcal{O}(\epsilon^2) = \mathcal{O}(\epsilon^2)$$

since $\bar{\epsilon}$ is on the surface,
while $\bar{F}_{const}$ is $\perp$ to the
surface ($\bar{F}_{cons}$ is the normal force here!)

For every holonomic system, it is always possible to prove that the forces of constraint do not do any work for a displacement which is consistent with the constraint.

$$\delta S = \mathcal{O}(\epsilon^2) \longrightarrow \text{The action is stationary}$$

Consequently, Hamilton principle's apply. However, we did not vary the path in an arbitrary way, rather we chose a variation that is consistent with the constraint, i.e. the path R is still a path on the constraint surface. This allows one to prove that Lagrange's equation with respect to the generalized coordinates are valid.

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2, t)\, dt$$

$$\underbrace{\delta S}_{\substack{\text{on the}\\ \text{surface}}} = 0 \longrightarrow \boxed{\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0}$$