Rotation about a fixed axis

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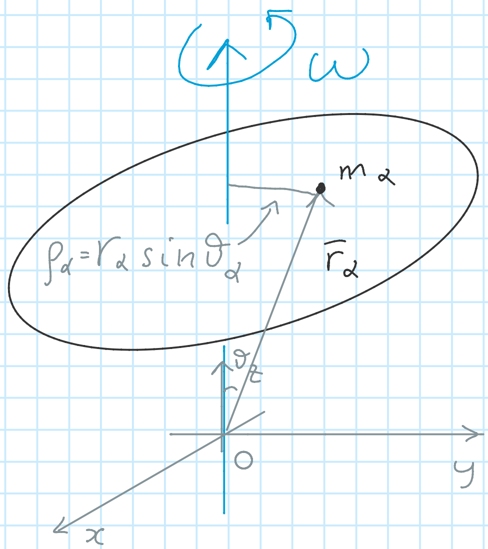
We established previously that the rotational motion of an object contributes to its kinetic energy.

In order to study the details of the rotational motion, it is convenient to start by considering the case of an object spinning with respect to an axis which is fixed in space. This situation is of course simpler than the more general case in which the object is spinning around an axis which is moving with translational and/or rotational motion in space (think for example about a spinning top).

$$\bar{\omega} = (0, 0, \omega)$$



One can start by calculating the angular momentum of the extended object by adding the angular momenta of each molecule in the object. For this study it is convenient to align the z axis to the axis of rotation and to calculate the angular momentum with respect to the origin O, located at some point along the z axis.

$$\bar{L} = \sum_{\alpha=1}^{N} \bar{\ell}_\alpha = \sum_\alpha \bar{r}_\alpha \times m_\alpha \bar{v}_\alpha$$

but

$$\bar{v}_\alpha = \underbrace{\bar{\omega} \times \bar{r}_\alpha}_{\substack{\text{angular speed} \\ \text{times distance from} \\ \text{the rotation axis}}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega \\ x_\alpha & y_\alpha & z_\alpha \end{vmatrix} = -\hat{\imath}\,\omega y_\alpha + \hat{\jmath}\,\omega x_\alpha$$

$$\bar{r}_\alpha \times \bar{v}_\alpha = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_\alpha & y_\alpha & z_\alpha \\ -\omega y_\alpha & \omega x_\alpha & 0 \end{vmatrix} = -\hat{i}\,\omega x_\alpha z_\alpha - \hat{j}\,\omega y_\alpha z_\alpha + \hat{k}\,\omega\left(x_\alpha^2 + y_\alpha^2\right)$$

$$\bar{\ell}_\alpha = m_\alpha \bar{r}_\alpha \times \bar{v}_\alpha = m_\alpha \omega \left[-x_\alpha z_\alpha \hat{i} - y_\alpha z_\alpha \hat{j} + (x_\alpha^2 + y_\alpha^2)\hat{k} \right]$$

At this stage it is convenient to consider first the z component of the angular momentum

$$L_z = \sum_\alpha \left(m_\alpha \bar{r}_\alpha \times \bar{v}_\alpha \right)_z = \sum_\alpha m_\alpha \omega \underbrace{\left(x_\alpha^2 + y_\alpha^2 \right)}_{\rho_\alpha^2}$$

$$= \underbrace{\left(\sum_\alpha m_\alpha \rho_\alpha^2 \right)}_{\equiv I_z} \omega = I_z \omega$$

MOMENT OF INERTIA ABOUT THE Z AXIS

The relation above establishes a link between the z component of the angular momentum and the angular velocity. The constant of proportionality is the moment of inertia about the z axis.

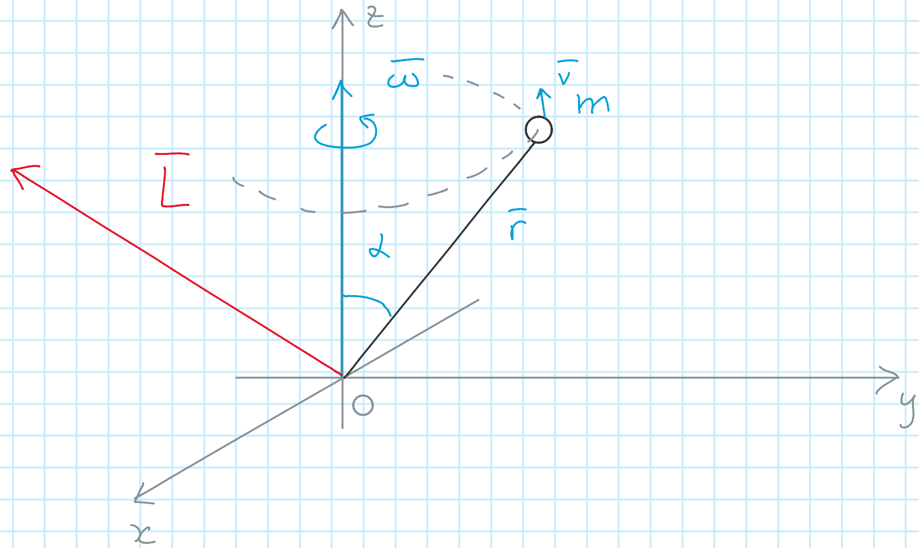
It must be observed that the x and y components of L are in general not equal to zero:

$$L_x = -\omega \sum_\alpha m_\alpha x_\alpha z_\alpha \qquad L_y = -\omega \sum_\alpha m_\alpha y_\alpha z_\alpha$$

Therefore, in the general case, the vector angular momentum is not aligned to the vector angular velocity

$$\bar{L} \neq I_z \bar{\omega}$$

A quick way to see that this can indeed be the case is to consider the object in the figure below, where only the end point of the bar as a significant mass



The direction of the angular momentum changes in time, so that there is a torque applied on the bar.

Finally, one can observe that the rotational kinetic energy can be written in terms of I and ω

$$T = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2 = \frac{1}{2}\sum_\alpha m_\alpha \rho_\alpha^2 \omega^2 = \frac{1}{2} I_z \omega^2$$

$\boxed{\text{Products of inertia}}$

One can observe that also the x and y components of L are proportional to ω. It is then convenient to introduce the following notation:

$$L_x = \left(-\sum_\alpha m_\alpha x_\alpha z_\alpha\right)\omega \equiv I_{xz}\,\omega$$

$$L_y = \left(-\sum_\alpha m_\alpha y_\alpha z_\alpha\right)\omega \equiv I_{yz}\,\omega$$

I_{xz} , I_{yz} PRODUCTS OF INERTIA

Furthermore, one can write

$$I_z = I_{zz} = \sum_\alpha m_\alpha \rho_\alpha^2 = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)$$

With this notation the total angular momentum can be written as

$$\bar{L} = I_{xz}\omega\hat{i} + I_{yz}\omega\hat{j} + I_{zz}\omega\hat{k}$$

Or, in the language of matrices

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} ? & ? & I_{xz} \\ ? & ? & I_{yz} \\ ? & ? & I_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

This is the first hint of the fact that one can define a tensor of inertia.