Effective potential and equivalent one dimensional problem

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So far we found Lagrange equations for the generalized coordinates r and phi. It is now easy to see that one can completely reduce the two body problem to a one dimensional problem for the coordinate r, provided that we introduce a suitably defined effective potential.

The Lagrange equations that we found are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\phi} = 0 \longrightarrow \mu r^2 \dot\phi = \underbrace{\ell}_{\text{constant}}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot r} - \frac{\partial \mathcal{L}}{\partial r} = 0 \longrightarrow \mu \ddot r = -\frac{dU}{dr} + \mu r \dot\phi^2 \qquad \equiv F_{cf} \text{ "centrifugal force"}$$

It is now convenient to use the first equation to get rid of the time derivative of Φ in the first equation

$$F_{cf} = \mu r \dot\phi^2 = \mu r \left(\frac{\ell}{\mu r^2}\right)^2 = \frac{\ell^2}{\mu r^3}$$

In turn, this can be written as a derivative with respect to r of a potential

$$F_{cf} = -\frac{d}{dr}\left(\underbrace{\frac{\ell^2}{2\mu r^2}}_{\equiv U_{cf}}\right)$$

Therefore the Lagrange equation for the variable r can be rewritten as

$$\mu \ddot r = -\frac{dU}{dr} - \frac{dU_{cf}}{dr} = -\frac{d}{dr}\Big[\underbrace{U(r) + \frac{\ell^2}{2\mu r^2}}_{\equiv U_{eff}}\Big]$$

EFFECTIVE POTENTIAL

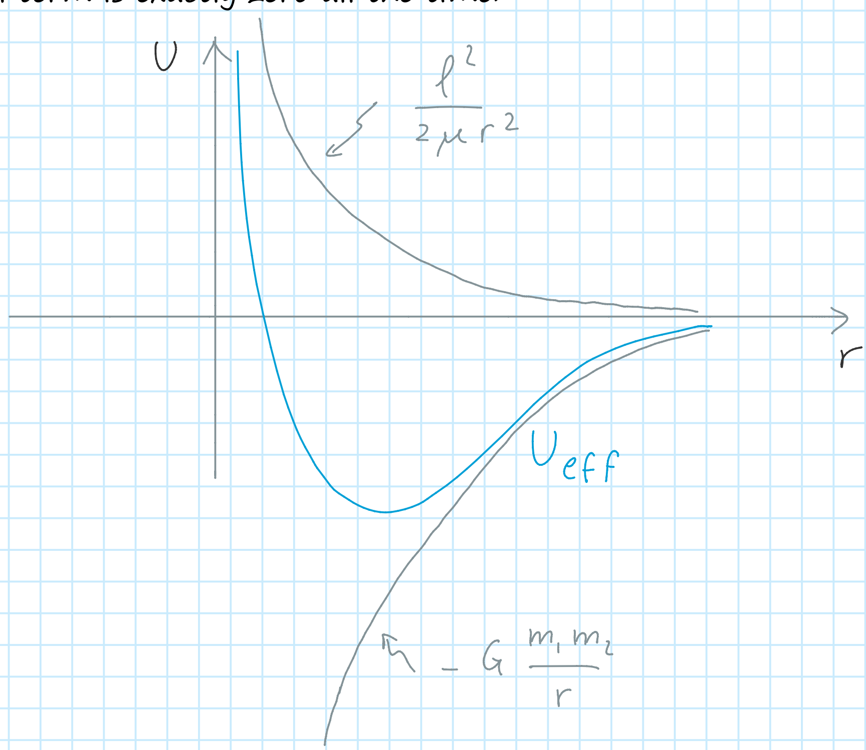
The two body problem was then completely reduced to a one dimensional problem, at the price of introducing an additional term in the potential.

Effective potential for a planet orbiting the sun

As a first application one can write down the effective potential for an object orbiting the sun.

$$\boxed{U_{eff}(r) = -G\frac{m_1 m_2}{r} + \frac{\ell^2}{2\mu r^2}} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

It is important to visualize the shape of the effective potential. At large r the effective potential is dominated by the gravitational term, while at small r the potential is dominated by the centrifugal term. Consequently, for large r the radial acceleration is negative, dominated by the gravitational force, while for small r the centrifugal "force" dominates and the radial acceleration is positive. A special case is encountered when the angular momentum l is exactly zero. In that case the centrifugal term is exactly zero all the time.



Conservation of energy

The system of two masses orbiting each other (and more in general in the two body problem) is an isolated system where the internal forces are conservative. Consequently, the mechanical energy of the system is conserved. This fact can be easily seen by rewriting Lagrange equation as follows

$$\mu \ddot{r} = \frac{1}{\dot{r}} \frac{d}{dt}\left(\frac{1}{2} m \dot{r}^2\right) = -\frac{d}{dr} U_{eff} = -\frac{d}{dt} U_{eff}(r) \underbrace{\frac{dt}{dr}}_{\frac{1}{\dot{r}}}$$

$$\frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2 + U_{eff}(r)\right) = 0$$

$$\frac{1}{2}\mu \dot{r}^2 + U_{eff}(r) = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

$$= \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\cancel{\mu^2}\, r^{\cancel{4}\,2} \dot{\phi}^2}{\cancel{\mu}\, \cancel{r^2}} + U(r)$$

$$= \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\phi}^2 + U(r) \equiv E$$

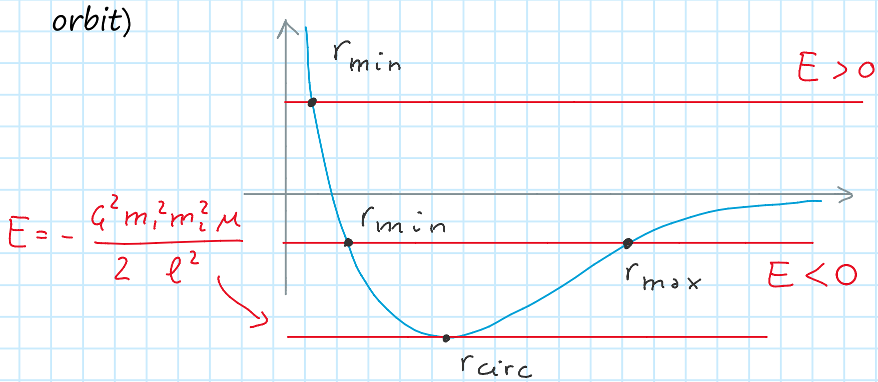
mechanical energy, conserved

Bounded and unbounded orbits

In the gravitational two body problem the conservation of energy alone is sufficient to explain a number of qualitative features. Notice that the effective potential was normalized in such a way that it goes to 0 when r goes to infinity. Obviously the total mechanical energy must be larger or equal to the effective potential, since the radial kinetic energy is positive

$$E = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r) \geq U_{eff}(r)$$

Therefore one can have two cases, $E > 0$ (unbounded orbit) and $E < 0$ (bounded orbit)



$E > 0 \longrightarrow r_{min} < r < \infty$ UNBOUNDED

$E < 0 \longrightarrow r_{min} < r < r_{max}$ BOUNDED

The minimum of the potential is located at

$$\frac{\partial U_{eff}}{\partial r} = 0 \longrightarrow G\frac{m_1 m_2}{r^2} - \frac{\ell^2}{\mu r^3} = 0$$

$$G m_1 m_2 = \frac{\ell^2}{\mu r} \qquad r_{min} = \frac{\ell^2}{G m_1 m_2 \mu}$$

$$\left[\frac{\ell^2}{G m_1 m_2 \mu}\right] = \frac{\cancel{kg}^2 \frac{m^{\cancel{2}}}{\cancel{s^2}} \cancel{m^2}}{\cancel{kg} \frac{\cancel{m}}{\cancel{s^2}} \frac{\cancel{m^2}}{\cancel{kg^2}} \cancel{kg}^{\cancel{3}\,2}} = m \quad \checkmark$$

The effective potential at the minimum is

$$U_{eff}(r_{min}) = -G m_1 m_2 \frac{G m_1 m_2 \mu}{\ell^2} + \frac{\cancel{\ell^2}}{2\cancel{\mu}} \frac{G^2 m_1^2 m_2^2 \mu^{\cancel{2}}}{\ell^{\cancel{4}\,2}}$$

$$= -\frac{1}{2} \frac{G^2 m_1^2 m_2^2 \mu}{\ell^2}$$

$$\left[\frac{G^2 m_1^2 m_2^2 \mu}{\ell^2}\right] = kg^{\cancel{2}} \frac{m^2}{s^{\cancel{4}\,2}} \frac{\cancel{m^4}}{\cancel{kg^4}} \frac{\cancel{kg^5}}{\cancel{kg^2} \frac{\cancel{m^4}}{\cancel{s^2}}} = kg \frac{m^2}{s^2} \quad \checkmark$$

If the energy is exactly equal to the effective potential at the minimum, then the radial velocity is zero and the orbit is circular, with radius r_min.