Precession of a top due to a weak torque

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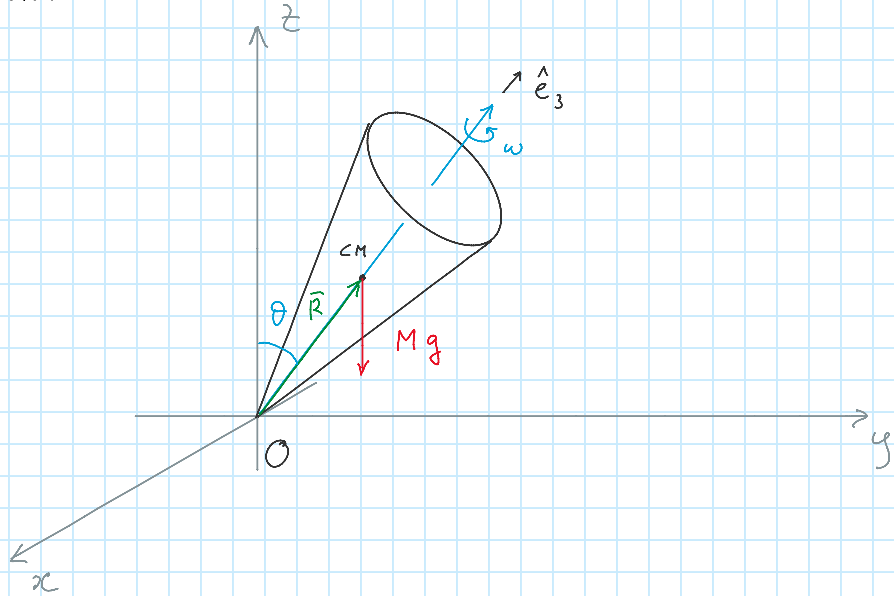
An interesting example of an object rotating around an axis that is not fixed in space is a spinning top. When the top is subject to a small (weak) torque, the axis of rotation has a precession motion around another axis. In order to analyze this situation let's consider the case shown in the figure below.



The inertia tensor calculated with respect to a frame centered in O, with the z axis along the axis of rotation of the cone is diagonal, because the axis of rotation is one of the principal axes of the object

$$\underline{I} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Therefore

$$\bar{\omega} \equiv \omega \hat{e}_3 \longrightarrow \bar{L} = \underline{I} \cdot \bar{\omega} = \lambda_3 \omega \hat{e}_3$$

If one imagines to switch gravity off one would see that there is no torque on the top and the top would keep on spinning around a fixed direction, indicated by the direction of the unit vector e_3 in the figure. However, gravity is present and applies a torque

$$\bar{\Gamma} = \bar{R} \times M \bar{g} \qquad\qquad \Gamma = R M g \sin\theta$$

Consequently, the time derivative of the angular momentum is not zero, and the time derivative of the vector ω is also not zero.

$$\bar{\Gamma} = \dot{\bar{L}} = \lambda_3 \dot{\bar{\omega}}$$

If ω changes in time, it will possibly acquire non zero components along the other two principal axes. However, if the torque is small, these components will remain small with respect to the component of the angular velocity along e_3. (this of course is a reasonable assumption but not a proof.) Under these conditions, the angular momentum remains aligned to the angular velocity and the moment and the two quantities remain linked by the relation

$$\bar{L} = \lambda_3 \, \omega \, \hat{e}_3$$

the direction of rotation will change in time

The torque Γ is perpendicular to the angular momentum, therefore L will change in direction but not in magnitude

$$\bar{\Gamma} = \dot{\bar{L}} \longrightarrow \bar{R} \times M \bar{g} = \lambda_3 \, \omega \, \dot{\hat{e}}_3$$

In addition, one can observe that

$$\bar{R} = R \hat{e}_3 \qquad\qquad \bar{g} = -g \hat{k}$$

$$\longrightarrow \quad \dot{\hat{e}}_3 = \frac{M g R}{\lambda \omega} \left(- \hat{e}_3 \times \hat{k} \right) = \underbrace{\frac{M g R}{\lambda \omega}}_{\bar{\Omega}} \hat{k} \times \hat{e}_3$$

The relation above indicates that the axis of the top rotates about the z axis with angular velocity Ω. Gravity causes the spinning top to precess slowly around a vertical cone that has an angular opening 2 θ.

Since the earth rotation axis is inclined at 23 degrees with respect to the plane that contains the earth's orbit around the sun, and the sun and the moon exert a small torque on the earth (due to the fact that the earth is not a perfect sphere) this precession motion affects the earth axes of rotation. It takes 26000 years to the earth axis to complete one revolution around a direction perpendicular to the orbit.