Principal axes of a cube about a corner

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In order to become familiar with the process that leads to identify the principal axes, one can conserved the case of the inertia tensor of a cube about one of the corners. When using axes parallel to the edges of the cube, the inertia tensor has the form

$$I = \mu \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

We previously established that the coefficient μ is

$$\mu = \frac{Ma^2}{12}$$

M = cube mass

a = cube side length

The matrix involved in the characteristic equation is therefore

$$I - \lambda \mathbb{1} = \begin{pmatrix} 8\mu - \lambda & -3\mu & -3\mu \\ -3\mu & 8\mu - \lambda & -3\mu \\ -3\mu & -3\mu & 8\mu - \lambda \end{pmatrix}$$

The corresponding characteristic equation is

$$\det(I - \lambda \mathbb{1}) = (8\mu - \lambda)\left[(8\mu - \lambda)^2 - 9\mu^2\right] + 3\mu\left[-9\mu^2 - 3\mu(8\mu - \lambda)\right]$$

$$- 3\mu\left[9\mu^2 + 3\mu(8\mu - \lambda)\right]$$

$$= (8\mu - \lambda)\left[(8\mu - \lambda)^2 - 9\mu^2\right] - 18\mu^2\left[3\mu + 8\mu - \lambda\right]$$

$$= (8\mu - \lambda)\left[(8\mu - \lambda) - 3\mu\right]\left[(8\mu - \lambda) + 3\mu\right] - 18\mu^2(11\mu - \lambda)$$

$$= (11\mu - \lambda)\left[(8\mu - \lambda)(5\mu - \lambda) - 18\mu^2\right]$$

$$= (11\mu - \lambda)\left[40\mu^2 - 13\mu\lambda + \lambda^2 - 18\mu^2\right]$$

$$= (11\mu - \lambda)\left[22\mu^2 - 13\mu\lambda + \lambda^2\right] = (11\mu - \lambda)^2(2\mu - \lambda)$$

Therefore

$$\det(I - \lambda \mathbb{1}) = 0 \longrightarrow \begin{cases} 11\mu - \lambda = 0 \longrightarrow \lambda_2 = \lambda_3 = 11\mu \\ \text{or} \\ 2\mu - \lambda = 0 \longrightarrow \lambda_1 = 2\mu \end{cases}$$

At this stage one can then solve the equation for the eigenvectors

$$(I - \lambda_1 \mathbb{1}) \cdot \overline{\omega} = \mu \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\boxed{\begin{aligned} 6\omega_x - 3\omega_y - 3\omega_z &= 0 \\ -3\omega_x + 6\omega_y - 3\omega_z &= 0 \\ -3\omega_x - 3\omega_y + 6\omega_z &= 0 \end{aligned}} \longrightarrow \begin{aligned} 2\omega_x - \omega_y - \omega_z &= 0 \quad (i) \\ -\omega_x + 2\omega_y - \omega_z &= 0 \quad (ii) \\ -\omega_x - \omega_y + 2\omega_z &= 0 \quad (iii) \end{aligned}$$

$$(i) - (ii) \longrightarrow \quad 3\omega_x - 3\omega_y = 0 \longrightarrow \omega_x = \omega_y$$

$$(i) \quad 2\omega_x - \omega_x - \omega_z = \omega_x - \omega_z = 0 \longrightarrow \omega_x = \omega_z$$

Consequently, the first principal axis is in the direction of the cube diagonal

$$\overline{\omega}_1 = (\hat{\imath} + \hat{\jmath} + \hat{k})\,\omega$$

$$\overline{L}_1 = I \cdot \overline{\omega}_1 = \underbrace{\frac{1}{6} M a^2}_{\text{moment of inertia of the cube about the diagonal}} \omega$$

For the second and third eigenvector one finds

$$\left(\mathbb{I} - \lambda \mathbb{1}\right) \cdot \overline{\omega} = \mu \begin{pmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

From the equation above one finds one single condition

$$\omega_x + \omega_y + \omega_z = 0$$

This condition alone does not allow one to uniquely identify the two other principal axes. However, one can see that the condition above implies that

$$\overline{\omega}_2 \cdot \overline{\omega}_1 = 0 \qquad \overline{\omega}_3 \cdot \overline{\omega}_1 = 0$$

Therefore any two directions, orthogonal among themselves and orthogonal to the first eigenvalue can be chosen as the other two principal axes. The freedom in the choice of these two axes is related to the fact that two eigenvalues are identical. When all of the eigenvalues are different the principal axes are determined in a unique way and there is no freedom in choosing them. If all of the eigenvalues are the same, any direction can be taken as a principal axis.

If one was to reevaluate the inertia tensor with respect to the new axes one would find a diagonal tensor

$$\mathbb{I}' = \frac{1}{12} M a^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$