Rotations about any axis

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It is not sufficient to discuss the rotation of an object about a single fixed Cartesian axis (that can be chosen as the z axis) for two reasons:

I) the axis of rotation of an object can change in time

II) we already observed that the direction of the angular momentum of an object and the direction of the angular velocity of an object do not always coincide. When the two directions coincide, the axis of rotation is a principal axis of the object. Each object has three, mutually perpendicular principal axes (this will be proved later). It turns out that the description of rotation is simpler when the principal axes are used as the reference frame. If this is done, the frame is chosen and the axis of rotation is in general pointing in a direction that is not one of the axes of the frame.

Angular momentum for an angular velocity pointing in an arbitrary direction

One typically faces one of two situations

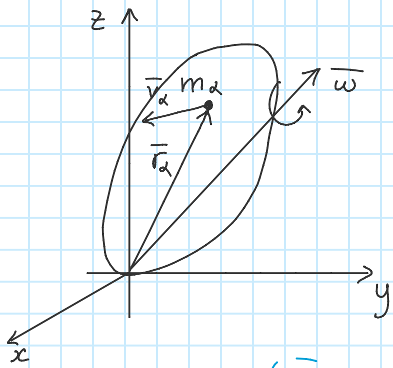
I) the spinning object has a fixed point, that it is natural to choose as the origin of the reference frame (ex. Spinning top). This does not imply that the axis of rotation is fixed in space, it can move as time goes by, but the fixed point does not move.

II) the object has no fixed point, but one can then describe the motion as the sum of the translational motion of the center of mass and the rotational motion of the object around the center of mass (ex. A drumstick thrown in the air). The center of mass is then taken as the origin of the reference frame.

In general the vector angular velocity at a given instant in time will have components along all of the reference frame directions.

$$\overline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

The total angular momentum of the body calculated with respect to the origin of the reference frame is then



$$\bar{L} = \sum_\alpha m_\alpha \bar{r}_\alpha \times \bar{v}_\alpha$$

$$= \sum_\alpha m_\alpha \bar{r}_\alpha \times (\bar{\omega} \times \bar{r}_\alpha)$$

<u>rem</u> $\quad \bar{a} \times (\bar{b} \times \bar{c}) = \bar{b}\,(\bar{a} \cdot \bar{c}) - \bar{c}\,(\bar{a} \cdot \bar{b})$

$$\bar{r} \times (\bar{\omega} \times \bar{r}) = \bar{\omega}\,(\bar{r} \cdot \bar{r}) - \bar{r}\,(\bar{\omega} \cdot \bar{r})$$

$$= \omega_x r^2 \hat{i} + \omega_y r^2 \hat{j} + \omega_z r^2 \hat{k} - x(\omega_x x + \omega_y y + \omega_z z)\hat{i}$$

$$- y(\omega_x x + \omega_y y + \omega_z z)\hat{j} - z(\omega_x x + \omega_y y + \omega_z z)\hat{k}$$

$$= \left[\omega_x (\cancel{x^2} + y^2 + z^2 - \cancel{x^2}) - \omega_y x y - \omega_z x z\right]\hat{i}$$

$$+ \left[\omega_y (x^2 + z^2) - \omega_x y x - \omega_z y z\right]\hat{j}$$

$$+ \left[\omega_z (x^2 + y^2) - \omega_x x z - \omega_y y z\right]\hat{k}$$

This can be written in the language of matrices

$$\bar{V} \equiv \bar{r} \times (\bar{\omega} \times \bar{r})$$

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -yx & z^2 + x^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

By inserting this result in the summation that gives the total angular momentum (and reactivating the subscript α that was neglected in order to keep the notation a bit lighter) one finds

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \underbrace{\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}}_{\text{INERTIA TENSOR}} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2)$$

$$I_{yy} = \sum_\alpha m_\alpha (x_\alpha^2 + z_\alpha^2)$$

$$I_{zz} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2)$$

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha \qquad I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$$

$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$$

$$I_{yx} = I_{xy} \qquad I_{zy} = I_{yz} \qquad I_{zx} = I_{xz}$$

THE TENSOR IS SYMMETRIC $\quad I_{ij} = I_{ji}$

Finally one can then write that

$$\boxed{L_i = \sum_{j=1}^{3} I_{ij}\, \omega_j}$$

$$\bar{L} = \underbrace{I}_{3\times 3 \text{ matrix}}\, \bar{\omega}$$