Charged particle in a magnetic field

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The goal of this section is to find a function L such that the corresponding Lagrange's equations reproduce Lorentz force

$$m\ddot{\bar{r}} = q\left(\bar{E} + \bar{v}\times\bar{B}\right)$$

LORENTZ FORCE

In order to achieve this goal, it is convenient to rewrite the fields in terms of the potentials (scalar and vector)

$$\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t} \qquad \bar{B} = \nabla\times\bar{A}$$

One can now prove that the Lagrangian is

$$\mathcal{L} = \frac{1}{2} m\dot{\bar{r}}^2 - qV + q\dot{\bar{r}}\cdot\bar{A}$$

$$= \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\left(V - \dot{x}A_x - \dot{y}A_y - \dot{z}A_z\right)$$

Indeed one can now check, component by component, that Lagrange's equations reproduce the components of Lorentz equation. Consider for example the x component

$$\frac{\partial\mathcal{L}}{\partial x} = -q\frac{\partial V}{\partial x} + q\dot{x}\frac{\partial A_x}{\partial x} + q\dot{y}\frac{\partial A_y}{\partial x} + q\dot{z}\frac{\partial A_z}{\partial x}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x} + qA_x$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = m\ddot{x} + q\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = m\ddot{x} + q\left(\frac{\partial A_x}{\partial t} + \cancel{\frac{\partial A_x}{\partial x}\dot{x}} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right)$$

$$+ q\left(\frac{\partial V}{\partial x} - \cancel{\dot{x}\frac{\partial A_x}{\partial x}} - \frac{\partial A_y}{\partial x}\dot{y} - \frac{\partial A_z}{\partial x}\dot{z}\right)$$

$$= m\ddot{x} + q\underbrace{\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right)}_{-E_x} + q\dot{y}\underbrace{\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)}_{-B_z}$$

$$+ q\dot{z}\underbrace{\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)}_{B_y} = 0$$

$$m\ddot{x} = q\left(E_x + \underbrace{\dot{y}B_z - \dot{z}B_y}_{\left[\dot{\bar{r}} \times \bar{B}\right]_x}\right)$$

The equation above is indeed the x component of Lorentz equation.

It is interesting to evaluate the generalized momentum coming from this Lagrangian

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + qA_x \longrightarrow \bar{p} = m\bar{v} + q\bar{A}$$

This is useful in quantum mechanics because the quantization rules require to replace the momentum with the operator

$$\bar{p} \xrightarrow{Q.M.} -i\hbar\nabla$$

Therefore $$m\bar{v} \xrightarrow{Q.M.} -i\hbar\nabla - q\bar{A}$$