Calculus of variations - introduction

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Many problems require the use of non cartesian coordinates, either because of the symmetry of the problem (ex. Spherical symmetry -> spherical coordinates, axial symmetry -> cylindrical coordinates) or because the object(s) that are in the problem are constrained to move on a surface or a line of a given shape.

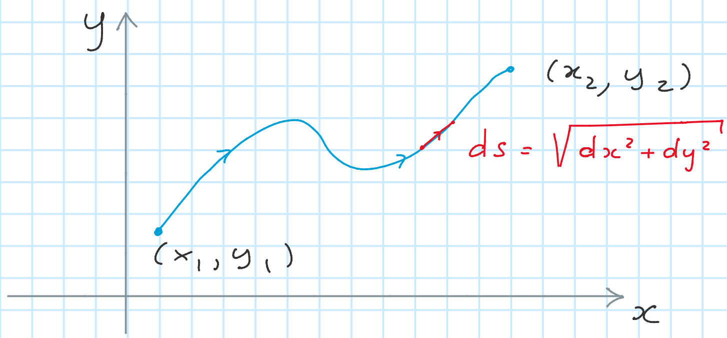
The equations of Newtonian mechanics become complicated very rapidly in non cartesian systems. It is useful to develop a formulation of mechanics that works equally well in different coordinate systems and that allows to account for constraints in an easier and "natural" way. This is the purpose of Lagrangian mechanics. Lagrangian mechanics is based upon the "variational principle". The latter is an idea that can be applied successfully in several domains in Physics.

Before we turn to the variational principle and we derive Lagrange's equations we need to become familiar with the calculus of variations

Calculus of variations

The main purpose of the calculus of variations is to figure out which integrand makes a given quantity, defined through an integral, maximum or minimum.

Example: Find the shortest path between two given points on a plane.



WHAT IS THE SHORTED PATH BETWEEN THE TWO POINTS?

OBVIOUSLY A STRAIGHT LINE

we want to prove this statement

$$y = y(x) \longrightarrow ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\underbrace{L}_{\text{LENGTH}} = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

What function $y(x)$ minimizes the length L?

Example: Fermat's principle

What is the path that light will take when going from an initial to a final point? If we stay in the same medium, the index of refraction stays the same and the answer is a straight path. However, when the path needs to cross boundary between two or more media, the answer is more complicated. Even more in general, the index of refraction can in principle change from point to point along the path. Pierre de Fermat (1601-1665, of "Fermat last theorem" fame) conjectured correctly that light takes the path that takes the shortest amount of time to travel from one point to another. One can formulate the problem mathematically as follows: The light travels in a medium with speed $v = ds/dt$. In addition $v = c/n$. Therefore $dt = ds/v = n/c\, ds$. Consequently, the total time of travel is

$$T = \int_1^2 dt = \int_1^2 \frac{n}{c}\, ds = \frac{1}{c}\int_1^2 n\, ds$$

$$= \frac{1}{c}\int_{x_1}^{x_2} n(x, y(x)) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If n is constant, the shortest time of travel will correspond to the shortest distance traveled. However, if n is not constant, then the shortest distance traveled will not correspond to the shortest time of travel.

We know that for a function, a necessary condition to have a stationary point (maximum, minimum or an inflection point with an horizontal tangent) is that the first derivative of the function vanishes (for an inflection point, also the second derivative will vanish.) This means that the function $f(x)$ does not change much if one moves x in an infinitesimal interval around the point in which $df/dx = 0$.

In the case in which the problem requires to find the stationary points of an integral, we need to ask ourselves what are the integrands that result in an integral that does not change much if the integrand is varied by an infinitesimal amount. The calculus of variation deals with how integrals change when the integrands are changed by an infinitesimal amount.