Motion of a spinning top

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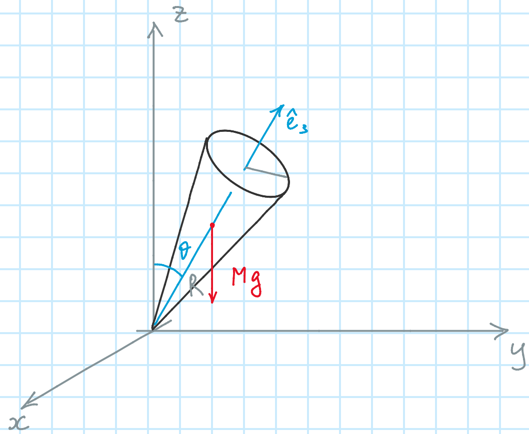
It is interesting to return to the motion of a spinning top and apply to it the language of the Euler angles and the Lagrangian approach.



The Lagrangian of the system will be

$$\mathcal{L} = \frac{1}{2}\lambda_1\left(\dot{\phi}^2\sin^2\vartheta + \dot{\vartheta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \dot{\phi}\cos\vartheta\right)^2 - MgR\cos\vartheta$$

The generalized coordinates in the Lagrangian are Euler's angles. One can then look at the three Lagrange equations that can be obtained from the Lagrangian. Notice that ψ and ϕ are ignorable coordinates.

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\vartheta}} - \frac{\partial\mathcal{L}}{\partial\vartheta} = 0$$

$$\lambda_1\ddot{\vartheta} - \Big[\lambda_1\dot{\phi}_1^2\sin\vartheta\cos\vartheta - \lambda_3\left(\dot{\psi} + \dot{\phi}\cos\vartheta\right)\dot{\phi}\sin\vartheta + MgR\sin\vartheta\Big] = 0$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \cancelto{0}{\frac{\partial\mathcal{L}}{\partial\phi}} = 0$$

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \lambda_1\dot{\phi} + \lambda_3\left(\dot{\psi} + \dot{\phi}\cos\vartheta\right)\cos\vartheta = L_z$$

The z component of the angular momentum is conserved. This is expected since the torque due to gravity does not have a component parallel to the z axis.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \cancelto{0}{\frac{\partial \mathcal{L}}{\partial \psi}} = 0$$

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \underbrace{\lambda_3 (\dot{\psi} + \dot{\phi}\cos\vartheta)}_{\text{component of } \vec{L} \text{ along } \hat{e}_3 = L_3} = \text{const.}$$

Also the fact that the component of angular momentum along the body axis is conserved is expected. Indeed the torque is also perpendicular to the body axis.

$\boxed{\text{Steady precession}}$

We want to investigate if a precession motion at fixed θ along the z axis is possible. It was already established that

$$\dot{\phi} = \frac{L_z - L_3\cos\vartheta}{\lambda_1 \sin\vartheta} \equiv \Omega$$

Therefore, if θ can be constant, then the time derivative of ϕ would also be constant. In other words if the precession motion describes a cone of angle 2θ along the z axis, then the angular velocity along the z axis is constant. This angular velocity can be determined from the Lagrange equation for θ.

$$\lambda_1 \ddot{\vartheta} = 0 = \lambda_1 \Omega^2 \sin\vartheta\cos\vartheta - \lambda_3 \underbrace{(\dot{\psi} + \dot{\phi}\cos\vartheta)}_{\omega_3} \Omega \sin\vartheta + M g R \sin\vartheta$$

$$\lambda_1 \Omega^2 \cos\vartheta - \lambda_3 \omega_3 \Omega + M g R = 0$$

$$\Omega = \frac{\lambda_3 \omega_3 \pm \sqrt{\lambda_3^2 \omega_3^2 - 4MgR\lambda_1 \cos\vartheta}}{2\lambda_1 \cos\vartheta}$$

If ω_3 is large the two roots are real

$$\omega_3^2 \gg \frac{4MgR\lambda_1 \cos\vartheta}{\lambda_3^2}$$

$$\Omega \simeq \frac{\lambda_3 \omega_3}{2\lambda_1 \cos\vartheta}\left[1 \pm \left(1 - \frac{\cancel{4}^{2} MgR\lambda_1 \cos\vartheta}{\cancel{2}\,\lambda_3^2 \omega_3^2} + \dots\right)\right]$$

$$\Omega_1 \simeq \frac{\cancel{\lambda_3} \cancel{\omega_3}}{\cancel{2} \cancel{\lambda_1} \cancel{\cos\vartheta}} \frac{\cancel{2} M g R \cancel{\lambda_1} \cancel{\cos\vartheta}}{\lambda_3^{\cancel{2}} \omega_3^{\cancel{2}}} = \underbrace{\frac{MgR}{\lambda_3 \omega_3}}$$

coincides with the precession due to a weak torque

$$\Omega_2 \simeq \frac{\lambda_3 \omega_3}{\lambda_1 \cos\vartheta} \qquad \Omega_2 \gg \Omega_1$$

Notice that the second larger precession velocity does not depend on g, so it should be present even in absence of gravity. Indeed it is exactly the free precession for a symmetric body that was found when considering Euler equations for zero torque.

Nutation

In general, while the top processes around the vertical axis, also the angle θ can vary. Thus the body axis can move closer and farther from the z axis in a motion called nutation. To study this motion it is convenient to rewrite the Lagrange equation for θ in terms of the conserved momenta

$$L_3 = \lambda_3(\dot\psi + \dot\phi\cos\vartheta)$$

$$L_z = \lambda_1 \dot\phi \sin^2\vartheta + L_3\cos\vartheta$$

$$\boxed{\dot\phi = \frac{L_z - L_3\cos\vartheta}{\lambda_1 \sin^2\vartheta}}$$

$$L_3 = \lambda_3\dot\psi + \lambda_3 \frac{L_z - L_3\cos\vartheta}{\lambda_1\sin^2\vartheta}\cos\vartheta$$

$$\boxed{\dot\psi = \frac{1}{\lambda_3}\left(L_3 - \frac{\lambda_3}{\lambda_1}\frac{L_z - L_3\cos\vartheta}{\sin^2\vartheta}\cos\vartheta\right)}$$

By inserting these expressions in the equation for θ one finds a regular second order differential equation that can be in principle be solved, at least numerically. However, it is possible to obtain qualitative information about the motion in an easier way. Indeed the total energy, written in terms of L_3 and L_z is

$$E = \frac{1}{2}\lambda_1\left(\dot\phi^2\sin^2\vartheta + \dot\vartheta^2\right) + \frac{1}{2}\lambda_3\left(\dot\psi + \dot\phi\cos\vartheta\right)^2 + MgR\cos\vartheta$$

$$= \frac{1}{2}\lambda_1\dot\vartheta^2 + \frac{\cancel{\lambda_1}}{2}\cancel{\sin^2\vartheta}\frac{(L_z - L_3\cos\vartheta)^2}{\lambda_1^{\cancel{2}}\sin^{\cancel{4}2}\vartheta}$$

$$+ \frac{\lambda_3}{2}\left[\frac{1}{\lambda_3}\left(L_3 - \frac{\lambda_3}{\lambda_1}\cancel{\frac{(L_z - L_3\cos\vartheta)}{\sin^2\vartheta}\cos\vartheta}\right)\right.$$

$$\left. + \cancel{\frac{L_z - L_3\cos\vartheta}{\lambda_1\sin\vartheta}\cos\vartheta}\right]^2 + MgR\cos\vartheta$$

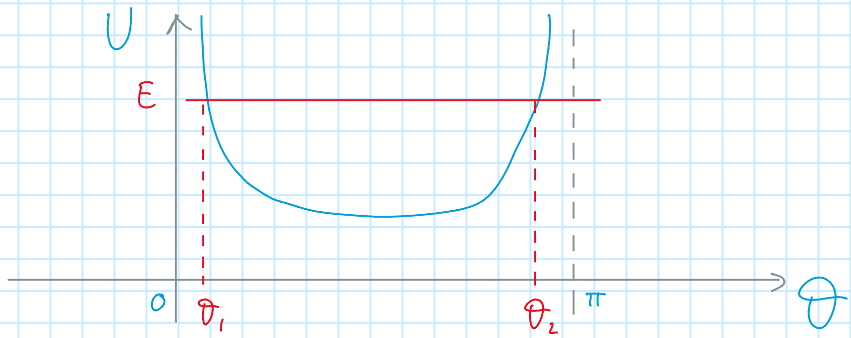
$$= \frac{1}{2}\lambda_1 \dot{\vartheta}^2 + \frac{(L_z - L_3\cos\vartheta)^2}{2\lambda_1 \sin^2\vartheta} + \frac{L_3^2}{2\lambda_3}$$

$$+ MgR\cos\vartheta$$

The problem is now one dimensional (the energy depends only on θ). One can then define an effective potential and then define an effective potential and plot it

$$U_{eff}(\vartheta) = \frac{(L_z - L_3\cos\vartheta)^2}{2\lambda_1 \sin^2\vartheta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\vartheta$$

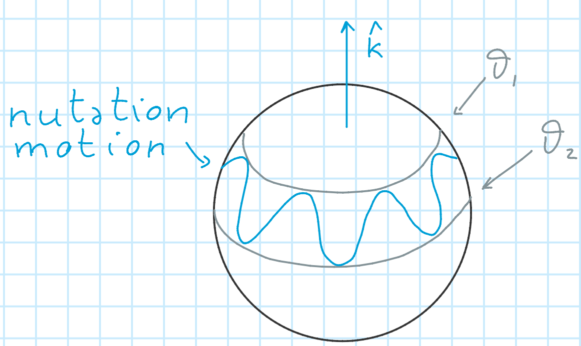
diverges at $\vartheta = 0$ and $\vartheta = \pi$



The angle θ oscillates back and forth between a minimum and a maximum. One can then look back at the time derivative of ϕ.

$$\dot{\phi} = \frac{L_z - L_3\cos\vartheta}{\lambda_1 \sin^2\vartheta}$$

if $|L_z| > |L_3|$ $\quad \dot{\phi} > 0$ always or $\dot{\phi} < 0$ always



if $|L_z| < |L_3|$ $\dot{\phi}$ can vanish for a specific ϑ $\longrightarrow$ if $\vartheta_1 < \vartheta < \vartheta_2$ then