Euler's equations

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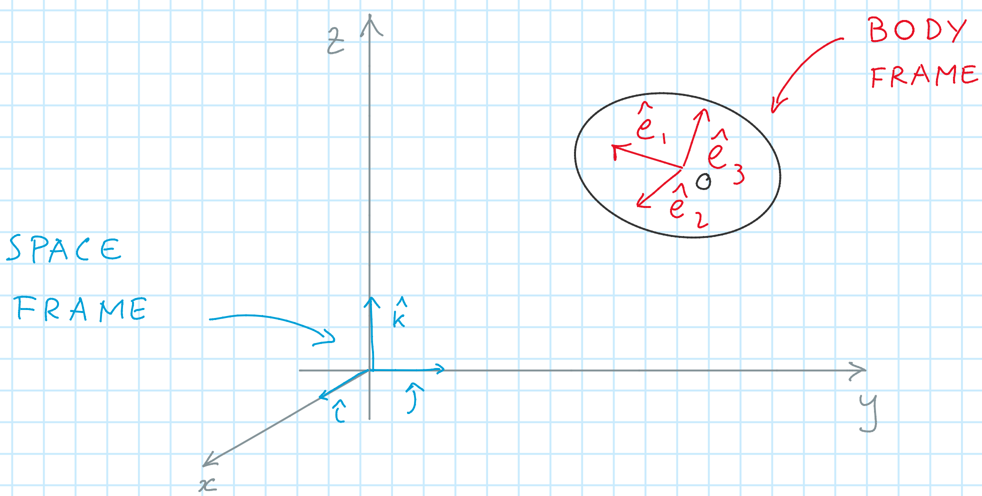
Here we want to derive Euler's equations, which apply to the rotation of an object about a fixed point or to the rotations of an object around its center of mass. In a certain sense these equations can be considered as the equivalent of Newton's second law for the case of rotational motion.

Since the inertia tensor is diagonal when calculated with respect to the principal axes of the body, we want to use the principal axes as our frame of reference. However, these axes are fixed in the rotating body. Consequently we need to deal with a rotating non inertial frame of reference. We need also to consider a second inertial frame in which the body is rotating. In this frame, Newton's laws of motion hold in their traditional form. The inertial frame is referred to as the space frame. The frame of the principal axes is called the body frame.



The angular momentum of the body, measured in the body frame is

$$\bar{L} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$$

With respect to the space frame instead, one can write the usual equation linking the time derivative of the angular momentum with the torque

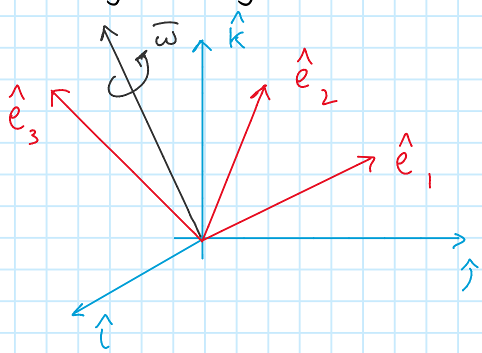
$$\left(\frac{d\bar{L}}{dt}\right)_{space} = \bar{\Gamma}$$

At this stage one needs to relate the time derivative of the angular momentum in the space frame with the time derivative of the angular momentum in the body frame. This relation is

$$\left(\frac{d\bar{L}}{dt}\right)_{space} = \left(\frac{d\bar{L}}{dt}\right)_{body} + \bar{\omega} \times \bar{L}$$

$\boxed{\text{Proof}}$

It is convenient to start by drawing the two frames in such a way that they have a common origin



$\bar{\omega}$ is the angular velocity of the body frame w.r.t the space frame

The angular momentum in the body frame can be written as

$$\bar{L} = L_1 \hat{e}_1 + L_2 \hat{e}_2 + L_3 \hat{e}_3 = \sum_{i=1}^{3} L_i \hat{e}_i$$

The time derivative in the body frame is therefore

$$\left(\frac{d\bar{L}}{dt}\right)_{body} = \sum_i \frac{dL_i}{dt} \hat{e}_i$$

If one wants the time derivative of L in the space frame, one should take into account the fact that in the space frame also the unit vectors e_i are changing in time, therefore

$$\left(\frac{d\bar{L}}{dt}\right)_{space} = \sum_{i=1}^{3} \frac{dL_i}{dt} \hat{e}_i + \sum_{i=1}^{3} L_i \left(\frac{d\hat{e}_i}{dt}\right)_{space}$$

The derivative in the second term in the equation above can be easily evaluated by observing that the body frame rotates with angular velocity ω in the space frame.

Consequently

$$\left(\frac{d\hat{e}_i}{dt}\right)_{space} = \bar{\omega} \times \hat{e}_i$$

$$\sum_i L_i \left(\frac{d\hat{e}_i}{dt}\right)_{space} = \sum_i L_i (\bar{\omega} \times \hat{e}_i) = \bar{\omega} \times \sum_i L_i \hat{e}_i = \bar{\omega} \times \bar{L}$$

So that finally one can conclude that

$$\left(\frac{d\bar{L}}{dt}\right)_{space} = \left(\frac{d\bar{L}}{dt}\right)_{body} + \bar{\omega} \times \bar{L}$$

Notice that this proof can be applied to any vector in the body frame, it is not specific to the case of the angular momentum.

Now one can introduce the relation above in Newton's second law for rotational motion

$$\left(\frac{d\bar{L}}{dt}\right)_{space} = \bar{\Gamma}$$

$$\boxed{\left(\frac{d\bar{L}}{dt}\right)_{body} + \bar{\omega} \times \bar{L} = \bar{\Gamma}}$$

EULER EQUATION

One can now write the equation above in components, taking the components with respect to the principals axes of the body.

$$\lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_1)\, \omega_2 \omega_3 = \Gamma_1$$

$$\lambda_2 \dot{\omega}_2 - (\lambda_3 - \lambda_1)\, \omega_3 \omega_1 = \Gamma_2$$

$$\lambda_3 \dot{\omega}_3 - (\lambda_1 - \lambda_2)\, \omega_1 \omega_2 = \Gamma_3$$

EULER EQUATIONS

In general, it is difficult to write the components of the torque w.r.t. a frame fixed in the body, but there are a few cases where the calculation is simple or manageable. A first example is the case in which there is no torque acting on the object. A second example is the spinning top, since the gravitational torque is always perpendicular to the axis of the top. In that case Γ_3 is always zero. In addition, because of the symmetry of the spinning cone, $\lambda_1 = \lambda_2$. Consequently the third of Euler's equations becomes

$$\lambda_3 \dot{\omega}_3 = 0$$

This proves that in the case of the spinning top the component of the angular velocity along the top axis is constant.